# WHAT ARE TENSORS?*

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Generally speaking, the content of present day college mathematics was established some two hundred years ago. We therefore find an ever widening gap between the mathematics that we teach and current creative work. The mere existence of this gap may not necessarily be deplorable. A theory has to undergo a process of ageing during which it is polished and simplified until it acquires that apparently immediate appeal which, for example, calculus and analytic geometry now have for our more gifted beginners. But the growing size of this gap should induce us to examine continually the possibility of revitalizing our curriculum.

Let us have a glance at analytic geometry. Its core consists of the theories of points, straight lines, and planes, of coordinate systems and their transformation, of conics in the plane and of quadrics in three-space. After the beginning of this century, these theories merged with certain algebraic theories dealing, for example, with systems of linear equations and with quadratic forms, into one new and powerful discipline, linear algebra. In the last twenty years linear algebra has developed rapidly. Becoming more and more abstract, it is covering an ever wider range of phenomena, and its applications reach into all branches of mathematics. We find an exposition of the present state of linear algebra in two French treatises, Bourbaki's *Algèbre Linéaire* of 1947 and his *Algèbre Multilinéaire* of 1948.

Geometry has fared poorly in this recent pursuit of abstraction and generality, and so the geometrical aspects of linear algebra are somewhat neglected. But I think analytic geometry could profit from the newer theory in two ways: (1) By making use of some of its ideas in the treatment of the standard subject matter, and (2) by incorporating some of its achievements in an advanced course. The latter would not necessarily have to lead the student through the latest abstractions of the theory of modules over rings. But it could give him a sound grounding in the now classical theory of finite dimensional vector spaces and its algebraic and geometric applications.

The chapter in this theory that we now discuss is the so-called tensor algebra. It is neglected in the textbooks on analytic geometry, and we have to turn to books on Riemannian geometry for an account of it. But it seems to me that it is bona fide analytic geometry and good geometry at that. Its most modern and abstract form is Bourbaki's multilinear algebra. And it is from the two Bourbaki treatises that we are borrowing what seems to us one of the fundamental ideas of modern analytic geometry, namely, the elimination of

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coordinates.†

We start with ordinary 3-space with its points $A, B, C, \cdots$. A segment $AB$ is determined by its end-points $A$ and $B$. By calling one of them, say $A$, the initial point of our segment and the other point $B$ its end-point, we arrive at the directed segment or "vector" $\overrightarrow{AB}$. Thus, every ordered pair of points $A, B$ determines a vector. If $A \neq B$, we may imagine the vector $\overrightarrow{AB}$ as an arrow leading from $A$ to $B$. Now we are not really interested in the initial point and the end-point of a vector, but only in its direction and its length. Thus, we shall call two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ equal if they have the same direction and the same length. Obviously this definition of equality satisfies the usual requirements, namely,

(1) Reflexivity: Every vector is equal to itself.

(2) Symmetry: If $\overrightarrow{AB}$ is equal to $\overrightarrow{CD}$, then $\overrightarrow{CD}$ is also equal to $\overrightarrow{AB}$.

(3) Transitivity: If two vectors are both equal to a third vector, then they are equal to one another.

Our definition of the equality of vectors enables us to move a given vector about, keeping it parallel to itself. In particular, any point may become its initial point. Its end-point is then determined.

We have neglected those vectors $\overrightarrow{AA}$ whose end-points coincide with their initial points. These vectors have the length zero and no specific direction. We identify all these vectors and consider them as a single vector, the "null-vector" $\mathfrak{o}$.

If two vectors $\mathfrak{a}$ and $\mathfrak{b}$ are given, we may choose the initial point $A$ of the first vector arbitrarily, say $\mathfrak{a} = \overrightarrow{AB}$. We now choose $B$ as the initial point of the second vector $\mathfrak{b}$. This determines its end-point $C$; thus $\mathfrak{b} = \overrightarrow{BC}$. The point $C$ is determined by the two vectors $\mathfrak{a}$ and $\mathfrak{b}$ and by the choice of the point $A$. The same will hold true of the vector $\mathfrak{c} = \overrightarrow{AC}$. In fact, $\mathfrak{c}$ depends only on the vectors $\mathfrak{a}$ and $\mathfrak{b}$. We write

$$\mathfrak{c} = \mathfrak{a} + \mathfrak{b} \quad \text{or} \quad \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}.$$

This vector addition has the following properties:

(I) Each pair of vectors $\mathfrak{a}, \mathfrak{b}$ determines one and only one third vector $\mathfrak{a}+\mathfrak{b}$.

(II) $\mathfrak{o}+\mathfrak{v}=\mathfrak{v}+\mathfrak{o}=\mathfrak{v}$ for every vector $\mathfrak{v}$.

Indeed, $\overrightarrow{AB}+\overrightarrow{BB}=\overrightarrow{AA}+\overrightarrow{AB}=\overrightarrow{AB}$. This states that the null-vector $\mathfrak{o}$ plays a role in vector addition analogous to that of the number 0 in ordinary addition.

† As late as twenty years ago, coordinates were considered the very essence of analytic geometry. An excellent textbook on this subject, published in 1931, contains the following observation: "It is the basic principle of analytic geometry to refrain from studying straight lines, planes, and space directly, and to examine the corresponding numerical manifolds instead. It is only afterwards that the algebraic results thus obtained are translated into results on manifolds of points." Cf. Schreier-Sperner, Analytische Geometrie und Algebra, vol. I, p. 8, Leipzig, 1931.

(III) To every vector $\mathfrak{a}$ there exists a vector denoted by $-\mathfrak{a}$ such that

$$\mathfrak{a} + (-\mathfrak{a}) = (-\mathfrak{a}) + \mathfrak{a} = \mathfrak{o}.$$

Obviously $\vec{AB}+\vec{BA}=\vec{BA}+\vec{AB}=\mathfrak{o}$. This states that every vector has an inverse.

(IV) $$\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c},$$

the associative law.

Finally, the commutative law,

(V) $$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}.$$

Any set of elements that satisfy these five conditions is called a commutative or abelian group. Thus, we may state: The vectors in 3-space form an abelian group.

There exists a second important operation on vectors, *i.e.*, their multiplication with real numbers. Let $\mathfrak{a}\neq\mathfrak{o}$, and let $\lambda$ be a real number. If $\lambda>0$, we define $\lambda\mathfrak{a}$ to be the vector whose direction is the same as that of $\mathfrak{a}$ and whose length is $\lambda$ times that of $\mathfrak{a}$. If $\lambda<0$, then $\lambda\mathfrak{a}$ is defined to be the vector whose direction is opposite to that of $\mathfrak{a}$ and whose length is $|\lambda|$ times the length of $\mathfrak{a}$. Finally, we define $0\cdot\mathfrak{a}=\mathfrak{o}$ and $\lambda\cdot\mathfrak{o}=\mathfrak{o}$. This multiplication satisfies the following additional rules:

(VI) $$1\cdot\mathfrak{a} = \mathfrak{a},$$

(VII) $$\lambda(\mu\mathfrak{a}) = (\lambda\mu)\cdot\mathfrak{a},$$

(VIII) $$(\lambda + \mu)\mathfrak{a} = \lambda\mathfrak{a} + \mu\mathfrak{a}, \qquad \lambda(\mathfrak{a} + \mathfrak{b}) = \lambda\mathfrak{a} + \lambda\mathfrak{b}.$$

Any set of elements that satisfy the conditions (I)–(VIII) is called a vector-space, or more accurately, a vector space over the real field. Thus, we have seen that the vectors in 3-space form such a vector space. A second example of a vector space is the set of all polynomials $f(x)$ in one variable $x$. Still another example is the set of all ordered $n$-tuples of real numbers $(a^1, \cdots, a^n)$ where $n$ is a fixed positive integer, and our vector operations are defined through

$$(a^1, a^2, \cdots, a^n) + (b^1, b^2, \cdots, b^n) = (a^1 + b^1, a^2 + b^2, \cdots, a^n + b^n)$$

and

$$\lambda(a^1, a^2, \cdots, a^n) = (\lambda a^1, \lambda a^2, \cdots, \lambda a^n).$$

It may not be very important that the vector space of all vectors in ordinary 3-space could be constructed without reference to coordinates. What does appear essential is this: once the concept of a vector space has been arrived at, it is studied using its definition rather than any coordinates that could be introduced. It is this axiomatic or conceptual approach that has led to the unification of theories that originally appeared far removed from each other.

The vectors in euclidean 3-space permit still another operation: Each pair

of vectors $\mathfrak{a}$, $\mathfrak{b}$ determines a real number $\mathfrak{ab}$, their scalar product. We can define it in the following fashion: $\mathfrak{a}^2$ is the square of the length of the vector $\mathfrak{a}$. We define $\mathfrak{ab}$ in such a way that if,

$$(\mathfrak{a} + \mathfrak{b})^2 = \mathfrak{a}^2 + \mathfrak{b}^2 + 2\mathfrak{ab},$$

that is

$$\mathfrak{ab} = \tfrac{1}{2}[(\mathfrak{a} + \mathfrak{b})^2 - \mathfrak{a}^2 - \mathfrak{b}^2]. \tag{1}$$

From the Pythagorean Theorem $\mathfrak{ab}=0$ if and only if either $\mathfrak{a}=\mathfrak{o}$ or $\mathfrak{b}=\mathfrak{o}$ or the vectors $\mathfrak{a}$ and $\mathfrak{b}$ are perpendicular. It is easy to see that the perpendicular projection of the vector $\mathfrak{a}$ on the vector $\mathfrak{b}$ is the vector

$$\frac{\mathfrak{ab}}{\mathfrak{b}^2}\cdot\mathfrak{b} \qquad [\mathfrak{b} \neq \mathfrak{o}]. \tag{2}$$

Our scalar product has the following properties:

$$\mathfrak{ab} = \mathfrak{ba}, \tag{IX}$$

$$(\lambda\mathfrak{a})\cdot\mathfrak{b} = \lambda\cdot\mathfrak{ab}, \tag{X}$$

$$(\mathfrak{a} + \mathfrak{b})\mathfrak{c} = \mathfrak{ac} + \mathfrak{bc}. \tag{XI}$$

The commutative law (IX) follows directly from (1). The other two rules can be interpreted geometrically. Thus (XI) states that the orthogonal projection of the vector $\mathfrak{a}+\mathfrak{b}$ on $\mathfrak{c}$ is equal to the sum of the projections of the vectors $\mathfrak{a}$ and $\mathfrak{b}$. Finally, our scalar product satisfies the following condition:

(XII) If $\mathfrak{ab}=\mathfrak{o}$ for a given vector $\mathfrak{a}$ and for every vector $\mathfrak{b}$, then $\mathfrak{a}=\mathfrak{o}$.

Suppose each pair of vectors $\mathfrak{a}$, $\mathfrak{b}$ of some vector space determines a real number $\mathfrak{ab}$ and suppose this mapping satisfies the rules (IX)–(XI). Then we call $\mathfrak{ab}$ the scalar product of the vectors $\mathfrak{a}$ and $\mathfrak{b}$. Any finite dimensional vector space can be provided with a scalar product in a great many different ways. We can introduce a scalar product into the vector space $R_n$ of all ordered $n$-tuples by defining, for example,

(i) $$(a^1, a^2, \cdots, a^n)\cdot(b^1, b^2, \cdots, b^n) = a^1b^1 + a^2b^2 + \cdots + a^nb^n,$$

or

(ii) $$(a^1, a^2, \cdots, a^n)\cdot(b^1, b^2, \cdots, b^n) = -a^1b^1 + a^2b^2 + \cdots + a^nb^n,$$

or

(iii) $$(a^1, a^2, \cdots, a^n)(b^1, b^2, \cdots, b^n) = a^2b^2 + \cdots + a^nb^n.$$

A vector space with a scalar product that also satisfies condition (XII) is sometimes called complete. Thus the vectors in euclidean 3-space form a complete vector-space. We can make $R_n$ complete by using the definitions (i) or (ii).

From now on we consider an arbitrary complete vector space $V$. It does not matter how we have obtained $V$ and its scalar product. All we need is the fact

that $V$ satisfies the axioms (I)–(XII). In particular, the dimension of $V$ need not be finite. We want to define tensors and tensor operations in $V$.

A simple tensor

$$\mathfrak{A} = \{\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_p\} \tag{3}$$

of order $p$ is an ordered $p$-tuple of vectors in $V$. Let

$$\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_p\} \tag{4}$$

be another simple tensor of order $p$. If none of the vectors $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_p$ is the null-vector, then we call $\mathfrak{A}$ and $\mathfrak{B}$ equal if corresponding vectors are parallel and if the products of their lengths are equal:

$$\mathfrak{A} = \mathfrak{B} \leftrightarrow \mathfrak{b}_1 = \lambda^1\mathfrak{a}_1, \mathfrak{b}_2 = \lambda^2\mathfrak{a}_2, \cdots, \mathfrak{b}_p = \lambda^p\mathfrak{a}_p$$

where

$$\lambda^1 \cdot \lambda^2 \cdot \cdots \cdot \lambda^p = 1.$$

We complete this definition by identifying all those simple tensors (3) of order $p$ for which at least one of the vectors $\mathfrak{a}_i$ is the null-vector. This simple tensor will be called the null-tensor $\mathfrak{O}_p$ of order $p$.

These definitions may be satisfactory from a geometric point of view, but they will lead to awkward proofs. We can overcome this difficulty by a more algebraic approach. Before defining the equality of the two simple tensors (3) and (4) of order $p$, we define their scalar product by means of

$$\mathfrak{A}\mathfrak{B} = \mathfrak{a}_1\mathfrak{b}_1 \cdot \mathfrak{a}_2\mathfrak{b}_2 \cdot \cdots \cdot \mathfrak{a}_p\mathfrak{b}_p. \tag{5}$$

And now we define:

(6) $\mathfrak{A} = \mathfrak{B} \leftrightarrow \mathfrak{A}\mathfrak{C} = \mathfrak{B}\mathfrak{C}$ for every simple tensor $\mathfrak{C}$ of order $p$.

It is not difficult to verify that these two definitions of equality are equivalent, that they satisfy the usual requirements, and that equal simple tensors have equal scalar products, that is,

(7) $\mathfrak{A} = \mathfrak{A}'$, $\mathfrak{B} = \mathfrak{B}'$ implies $\mathfrak{A}\mathfrak{B} = \mathfrak{A}'\mathfrak{B}'$.

We want to extend our vector operations to tensors. The product of a simple tensor with a scalar, *i.e.*, number can be readily defined:

$$\lambda\{\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_p\} = \{\lambda\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_p\}.$$

Since the latter tensor is equal to

$$\{\mathfrak{a}_1, \lambda\mathfrak{a}_2, \cdots, \mathfrak{a}_p\} = \cdots = \{\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{p-1}, \lambda\mathfrak{a}_p\},$$

the lack of symmetry in this definition is only apparent.

In order to define addition, we have to introduce more general tensors. A tensor of order $p$ is a formal finite sum of simple tensors of order $p$:

$$\mathfrak{A} = \{\mathfrak{a}_1^{(1)}, \mathfrak{a}_2^{(1)}, \cdots, \mathfrak{a}_p^{(1)}\} + \cdots + \{\mathfrak{a}_1^{(r)}, \mathfrak{a}_2^{(r)}, \cdots, \mathfrak{a}_p^{(r)}\}. \tag{8}$$

Let

$$\mathfrak{B} = \{\mathfrak{b}_1^{(1)}, \mathfrak{b}_2^{(1)}, \cdots, \mathfrak{b}_p^{(1)}\} + \cdots + \{\mathfrak{b}_1^{(s)}, \mathfrak{b}_2^{(s)}, \cdots, \mathfrak{b}_p^{(s)}\} \tag{9}$$

be another tensor of the same order. Then we define their scalar product through

$$\begin{aligned}\mathfrak{A}\mathfrak{B} &= \{\mathfrak{a}_1^{(1)}, \mathfrak{a}_2^{(1)}, \cdots, \mathfrak{a}_p^{(1)}\}\{\mathfrak{b}_1^{(1)}, \mathfrak{b}_2^{(1)}, \cdots, \mathfrak{b}_p^{(1)}\} + \cdots \\ &\quad + \{\mathfrak{a}_1^{(r)}, \mathfrak{a}_2^{(r)}, \cdots, \mathfrak{a}_p^{(r)}\}\{\mathfrak{b}_1^{(s)}, \mathfrak{b}_2^{(s)}, \cdots, \mathfrak{b}_p^{(s)}\} \\ &= \sum_{\rho=1}^{r}\sum_{\sigma=1}^{s}\{\mathfrak{a}_1^{(\rho)}, \mathfrak{a}_2^{(\rho)}, \cdots, \mathfrak{a}_p^{(\rho)}\}\{\mathfrak{b}_1^{(\sigma)}, \mathfrak{b}_2^{(\sigma)}, \cdots, \mathfrak{b}_p^{(\sigma)}\} \\ &= \sum_{\rho=1}^{r}\sum_{\sigma=1}^{s}\sum_{\tau=1}^{p}\mathfrak{a}_\tau^{(\rho)}\mathfrak{b}_\tau^{(\sigma)}.\end{aligned}$$

Thus each summand of $\mathfrak{A}$ is multiplied with each summand of $\mathfrak{B}$. The equality of tensors of order $p$ can now be defined in a manner similar to (6): $\mathfrak{A}=\mathfrak{B}\leftrightarrow\mathfrak{A}\mathfrak{C}=\mathfrak{B}\mathfrak{C}$ for every tensor $\mathfrak{C}$ of order $p$. We can justify this definition by showing that it satisfies the usual requirements, that it is equivalent to the previous definitions in the case of simple tensors and that finally equal tensors have equal scalar products, *i.e.*, (7) holds.

It is now obvious how we have to define the addition of tensors of the same order $p$ and the multiplication of tensors by scalars; namely, if $\mathfrak{A}$ and $\mathfrak{B}$ are given by (8) and (9), then $\lambda\mathfrak{A}$ and $\mathfrak{A}+\mathfrak{B}$ are defined to be

$$\begin{aligned}\lambda\mathfrak{A} &= \{\lambda\mathfrak{a}_1^{(1)}, \mathfrak{a}_2^{(1)}, \cdots, \mathfrak{a}_p^{(1)}\} + \cdots + \{\lambda\mathfrak{a}_1^{(r)}, \mathfrak{a}_2^{(r)}, \cdots, \mathfrak{a}_p^{(r)}\}, \\ \mathfrak{A}+\mathfrak{B} &= \{\mathfrak{a}_1^{(1)}, \mathfrak{a}_2^{(1)}, \cdots, \mathfrak{a}_p^{(1)}\} + \cdots + \{\mathfrak{a}_1^{(r)}, \mathfrak{a}_2^{(r)}, \cdots, \mathfrak{a}_p^{(r)}\} \\ &\quad + \{\mathfrak{b}_1^{(1)}, \mathfrak{b}_2^{(1)}, \cdots, \mathfrak{b}_p^{(1)}\} + \cdots + \{\mathfrak{b}_1^{(s)}, \mathfrak{b}_2^{(s)}, \cdots, \mathfrak{b}_p^{(s)}\}.\end{aligned}$$

Again, we can verify that equal tensors have equal multiples, and that sums of equal tensors are equal, that is,

$$\mathfrak{A} = \mathfrak{A}', \mathfrak{B} = \mathfrak{B}' \quad \text{implies} \quad \lambda\mathfrak{A} = \lambda\mathfrak{A}' \quad \text{and} \quad \mathfrak{A}+\mathfrak{B} = \mathfrak{A}'+\mathfrak{B}'.$$

We have constructed a set of elements, the tensors of order $p$, and we have introduced three operations into this set: multiplication with scalars, addition, and scalar products. These were the same operations we used in the definition of the complete vector space from which we started. In fact, it can be proved that the tensors of order $p$ form a complete vector space $V_p$. This space depends on our original vector space $V$ and on the value of $p$. Thus $V_1$ differs from $V$ in nothing but its notation. It therefore may be identified with $V$. The real numbers are called tensors of order zero.

So far, the various $V_p$'s are connected with each other only by their common dependence on $V$. However, there are two operations on tensors that tie these $V_p$'s together, as indicated in the following. The general product of the

tensor (8) of order $p$ by a tensor

$$\mathfrak{C} = \sum_{\tau=1}^{t} \{\mathfrak{c}_1^{(\tau)}, \mathfrak{c}_2^{(\tau)}, \cdots, \mathfrak{c}_q^{(\tau)}\}$$

of order $q$ is the following tensor of order $p+q$:

$$\{\mathfrak{A}, \mathfrak{C}\} = \sum_{\rho=1}^{r} \sum_{\tau=1}^{t} \{\mathfrak{a}_1^{(\rho)}, \mathfrak{a}_2^{(\rho)}, \cdots, \mathfrak{a}_p^{(\rho)}, \mathfrak{c}_1^{(\tau)}, \mathfrak{c}_2^{(\tau)}, \cdots, \mathfrak{c}_q^{(\tau)}\}.$$

Again, general products of equal tensors are equal, i.e.,

$$\mathfrak{A} = \mathfrak{A}', \ \mathfrak{C} = \mathfrak{C}' \quad \text{implies} \quad \{\mathfrak{A}, \mathfrak{C}\} = \{\mathfrak{A}', \mathfrak{C}'\}.$$

Let $p \geqq 2$. We contract the tensor $\mathfrak{A}$ of order $p$ by first selecting two indices $\alpha, \beta$, say $1 \leqq \alpha < \beta \leqq p$, and then forming the new tensor

$$\mathfrak{A}^* = \sum_{\rho=1}^{r} \mathfrak{a}_\alpha^{(\rho)} \mathfrak{a}_\beta^{(\rho)} \{\mathfrak{a}_1^{(\rho)}, \cdots, \mathfrak{a}_{\alpha-1}^{(\rho)}, \mathfrak{a}_{\alpha+1}^{(\rho)}, \cdots, \mathfrak{a}_{\beta-1}^{(\rho)}, \mathfrak{a}_{\beta+1}^{(\rho)}, \cdots, \mathfrak{a}_p^{(\rho)}\}.$$

Thus the $\alpha$th and the $\beta$th vector are taken out of each brace. The contracted tensor $\mathfrak{A}^*$ has the order $p-2$. It will, as a rule, depend on the choice of the indices $\alpha$ and $\beta$. But it is important that equal tensors contracted in the same fashion yield equal tensors.

We have developed tensor algebra for arbitrary complete vector spaces $V$. If the dimension of $V$ is finite, then we can choose a basis of $V$ and introduce covariant and contravariant components of a vector. After this we can define the various types of components of a tensor and follow our tensor operations in terms of these components. We then arrive at the standard formulas we find in any textbook on tensor analysis or Riemannian geometry.

## Appendix

The reader may be interested in a few remarks on some topics not mentioned in the above discussion.

**1. Tensors over finite dimensional vector spaces.** Let $V$ be an $n$-dimensional complete vector space, and let

$$\mathfrak{B} \equiv \mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_n$$

be a basis of $V$. Put $g_{ik} = \mathfrak{b}_i \mathfrak{b}_k$.† Then the matrix $(g_{ik})$ is regular and has an inverse $(g^{ik})$. Define

† Thus each of the numbers $g_{ik}$ is the scalar product of the corresponding vectors $\mathfrak{b}_i$ and $\mathfrak{b}_k$. The last formula of this section shows that the $g_{ik}$ are the covariant components of the fundamental tensor $\mathfrak{G}$. It should be noted however that usually the *set* of numbers $g_{ik}$ itself is called a covariant tensor.

The basis $\mathfrak{B}^1$ and hence the various types of tensor components can be constructed without referring to the $g_{ik}$ and $g^{ik}$: The vector $\mathfrak{b}^k$ is the one and only vector whose scalar product with $\mathfrak{b}_k$ is equal to one, while its scalar products with the other $\mathfrak{b}_i$'s vanish.

$$\mathfrak{b}^k = \sum_1^n \lambda\, g^{k\lambda}\mathfrak{b}_\lambda.$$

The vectors $\mathfrak{b}^1, \mathfrak{b}^2, \cdots, \mathfrak{b}^n$ form another basis $\mathfrak{B}^1$ of $V$.

By means of $\mathfrak{B}$ and $\mathfrak{B}^1$ we can readily construct bases of the vector space $V_p$ of all tensors of order $p$ over $V$. Thus either of the following sets of simple tensors form such a basis:

$$\begin{aligned} &\mathfrak{B}_p\colon \quad \{\mathfrak{b}_{i_1}, \mathfrak{b}_{i_2}, \cdots, \mathfrak{b}_{i_p}\} \\ &\mathfrak{B}^p\colon \quad \{\mathfrak{b}^{i_1}, \mathfrak{b}^{i_2}, \cdots, \mathfrak{b}^{i_p}\} \end{aligned} \qquad [i_1, i_2, \cdots, i_p = 1, 2, \cdots, n].$$

In particular $V_p$ has the dimension $n^p$. The $p$-times contravariant components $p^{i_1, i_2, \ldots, i_p}$ of a tensor $\mathfrak{T}$ with respect to $\mathfrak{B}$ are its contravariant components if we consider $\mathfrak{T}$ as an element of the vector space $V_p$ with the basis $\mathfrak{B}_p$. Thus

$$\mathfrak{T} = \sum_{i_1, i_2, \ldots, i_p}^{n} P^{i_1, i_2, \cdots, i_p}\{\mathfrak{b}_{i_1}, \mathfrak{b}_{i_2}, \cdots, \mathfrak{b}_{i_p}\}.$$

Similarly, the $p$-times covariant components $P_{i_1 i_2, \ldots, i_p}$ of $\mathfrak{T}$ with respect to $\mathfrak{B}$ are its covariant components with respect to $\mathfrak{B}_p$. Thus

$$P_{i_1, i_2, \ldots, i_p} = \mathfrak{T} \cdot \{\mathfrak{b}_{i_1}, \mathfrak{b}_{i_2}, \cdots, \mathfrak{b}_{i_p}\}.$$

The $P^{i_1, i_2, \ldots, i_p}$ and $P_{i_1 i_2, \ldots, i_p}$ are respectively the covariant and contravariant components of $\mathfrak{T}$ with respect to the basis $\mathfrak{B}^p$. The mixed components of $\mathfrak{T}$ can be introduced in a similar fashion, for example,

$$P_{i_1}^{\cdot i_2, \cdots, i_p} = \mathfrak{T} \cdot \{\mathfrak{b}_{i_1}, \mathfrak{b}^{i_2}, \cdots, \mathfrak{b}^{i_p}\}$$

and

$$\mathfrak{T} = \sum_{i_1, i_2, \ldots, i_p}^{n} P_{i_1}^{\cdot i_2, \cdots, i_p}\{\mathfrak{b}^{i_1}, \mathfrak{b}_{i_2}, \cdots, \mathfrak{b}_{i_p}\}.$$

Instead of introducing the basis $\mathfrak{B}^1$ most textbooks prefer working with the fundamental tensor

$$\mathfrak{G} = \sum_{i,k=1}^{n} g^{ik}\{\mathfrak{b}_i, \mathfrak{b}_k\} = \sum_{i,k=1}^{n} g_{ik}\{\mathfrak{b}^i, \mathfrak{b}^k\} = \sum_{i=1}^{n} \{\mathfrak{b}_i, \mathfrak{b}^i\}.$$

$\mathfrak{G}$ is independent of the choice of the basis $\mathfrak{B}$.

**2. Multivectors.** Let $V$ be any complete vector space. Interesting subspaces of $V_p$ are formed by those tensors of order $p$ that satisfy suitable symmetry conditions. The following example may suffice:

We map each tensor

$$\mathfrak{T} = \sum \{\mathfrak{t}_1^{(\lambda)}, \mathfrak{t}_2^{(\lambda)}, \cdots, \mathfrak{t}_p^{(\lambda)}\}$$

on $V_p$ on the tensor

$$A\mathfrak{T} = \frac{1}{p!}\sum_{\lambda}\sum_{(i_1,i_2,\cdots,i_p)} \pm \{\mathfrak{t}_{i_1}^{(\lambda)}, \mathfrak{t}_{i_2}^{(\lambda)}, \cdots, \mathfrak{t}_{i_p}^{(\lambda)}\}.$$

The inner sum is extended over all the permutations $(i_1, i_2, \cdots, i_p)$ of the numbers 1, 2, $\cdots$, $p$, and the plus sign holds for even permutations, the minus sign for the odd ones. This mapping

$$A\colon\ \mathfrak{T} \to A\mathfrak{T}$$

of $V_p$ into itself is linear; more accurately, it is a projection of $V_p$ on a subspace $A\,V_p$; that means, a tensor $\mathfrak{V}$ lies in $A\,V_p$ if and only if $\mathfrak{V} = A\mathfrak{V}$. A tensor in $A\,V_p$ which is the image of a simple tensor is called a $p$-vector. Thus, every tensor in $A\,V_p$ is a sum of $p$-vectors. The subspace $A\,V_p$ is again complete.

Since both $p$-vectors and determinants of order $p$ are invariant under the alternating group of order $p$, their theories are closely related. Only a few results can be quoted:

Let $\mathfrak{V}$ and $\mathfrak{W}$ be two $p$-vectors, say

$$\mathfrak{V} = A\{\mathfrak{v}_1, \cdots, \mathfrak{v}_p\} \quad \text{and} \quad \mathfrak{W} = A\{\mathfrak{w}_1, \cdots, \mathfrak{w}_p\}.$$

Then

$$\mathfrak{V}\mathfrak{W} = \frac{1}{p!}\begin{vmatrix} \mathfrak{v}_1\mathfrak{w}_1 & \cdots & \mathfrak{v}_1\mathfrak{w}_p \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \mathfrak{v}_p\mathfrak{w}_1 & \cdots & \mathfrak{v}_p\mathfrak{w}_p \end{vmatrix}.$$

$\mathfrak{V}$ is equal to the null-tensor [cf. p. 301] if and only if the vectors $\mathfrak{v}_1, \mathfrak{v}_2, \cdots, \mathfrak{v}_p$ are linearly dependent. If $\mathfrak{V} \neq \mathfrak{O}_p$, then $\mathfrak{V} = \mathfrak{W}$ if and only if the $\mathfrak{w}_i$'s are linear combinations of the $\mathfrak{v}_k$'s with determinant 1, *i.e.*, if the two parallelotopes determined by the vectors $\mathfrak{v}_1, \mathfrak{v}_2, \cdots, \mathfrak{v}_p$, respectively, $\mathfrak{w}_1, \mathfrak{w}_2, \cdots, \mathfrak{w}_p$ span the same $p$-space and are equal in volume and orientation.

Let $\mathfrak{S}$ and $\mathfrak{T}$ be tensors of order $p$, respectively $q$ and let $\{\mathfrak{S}, \mathfrak{T}\}$ denote their general product. Then

$$A\{\mathfrak{S}, \mathfrak{T}\} = A\{A\mathfrak{S}, A\mathfrak{T}\}.$$

This sum of $(p+q)$-vectors is called the exterior product of $A\mathfrak{S}$ by $A\mathfrak{T}$. Obviously, the exterior product of two multivectors is again a multivector.

Finally, given a $p$-vector $\mathfrak{V}$ and a $q$-vector $\mathfrak{W}$; $q < p$. Then there exists one and only one sum $\mathfrak{V}\mathfrak{W}$ of $(p-q)$-vectors such that

$$\mathfrak{V} \cdot A\{\mathfrak{W}, \mathfrak{U}\} = \mathfrak{V}\mathfrak{W} \cdot \mathfrak{U}$$

for every $(p-q)$-vector $\mathfrak{U}$. Up to a numerical factor, this "inner product" $\mathfrak{V}\mathfrak{W}$ is equal to the $q$-times contracted general product $\{\mathfrak{V}, \mathfrak{W}\}$.

# GEOMETRICAL EXTREMA SUGGESTED BY A LEMMA OF BESICOVITCH

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**1. Introduction.** In [1] Besicovitch needed as a lemma a result of the following type.

THEOREM 1. *Given a set* $\Gamma$ *of coplanar circles, the center of no one of them being in the interior of another, and* $U$ *the circle (or a circle) of* $\Gamma$ *whose radius does not exceed the radius of any other circle of* $\Gamma$, *then the number of circles meeting* $U$ *does not exceed* 18.

Besicovitch proved the weaker theorem obtained from this one by replacing 18 by 21. In this paper we shall prove Theorem 1 as it stands. The number 18 cannot be replaced by a smaller number, as is shown by the example in which all of the circles have radius 1 and the centers are at the following points in a polar coordinate system: the origin, the points $(1, h\cdot 60°)$ where $h=0, 1, \cdots, 5$, and the points $(2\cos 15°, (2k+1)\cdot 15°)$ where $k=0, 1, \cdots, 11$.

We prove Theorem 1 by establishing its equivalence with Theorem 2 and then proving the latter.

THEOREM 2. *It is impossible to have* 20 *points in* a circle of radius* 2 *such that one of the points is at the center and all of the mutual distances are at least* 1.

Naturally one can ask the general question: For any positive integer $n$, what is the radius $r(n)$ of the smallest† circle containing $n$ points one of which is at the center and all the mutual distances between which are at least 1? Thus Theorem 2 says that $r(20)>2$. A related question is the following: Of all sets of $n$ points in the plane such that the mutual distances are all at least 1 (with no restriction on the arrangement of the points) what set has the minimum diameter‡ $D(n)$? The following theorem answers this question for $n=7$.

THEOREM 3. *A set of seven points in the plane whose mutual distances are all at least* 1 *has diameter at least* 2, *with this value being attained only by the set of points consisting of the vertices and circumcenter of a regular hexagon of side-length* 1.

The asymptotic behavior of $r(n)$ and $D(n)$ is well-known§; in fact, for large $n$ the regular hexagonal lattice gives about the best results, so that $D(n)\sim 2r(n)\sim(12/\pi^2)^{1/4}n^{1/2}$ as $n\to\infty$. However we are interested here in small values of $n$.

* The term "in" is supposed to include the boundary.

† It is not difficult to see that the greatest lower bound is attained here.

‡ The diameter of a set of points is the least upper bound of all the mutual distances. For a finite set this is simply the greatest mutual distance. Also note that the diameter of a polygon is equal to the diameter of the point-set consisting of the vertices.

§ See [3], [4], [5], [6].

For each $n$ the values of $r(n)$ and $D(n)$ can be calculated to any desired degree of accuracy by constructing a sufficiently fine mesh, but the proofs of our theorems will show that to accomplish this practically is not easy.

The first few values of $r(n)$ are as follows: $r(2)=r(3)=\cdots=r(7)=1$, $r(8)=\frac{1}{2}$ cosec $(180°/7)=1.15\cdots$, $r(9)=\frac{1}{2}$ cosec $22°5=1.30\cdots$, $r(10)=\frac{1}{2}$ cosec $20°=1.46\cdots$, $r(11)=\frac{1}{2}$ cosec $18°=1.61\cdots$, $r(12)=1.68\cdots$, and $r(13)=\sqrt{3}=1.73\cdots$. The evaluation of $r(8)$ through $r(11)$ will come out in the proof of Theorem 2; the evaluation of $r(12)$ and $r(13)$ can be effected by similar methods. Further, the example before the statement of Theorem 2 shows that $r(19)\leqq 2\cos 15°=1.93\cdots$. Also $r(20)>2$.

The first few values of $D(n)$ are also easy to find: $D(2)=D(3)=1$, $D(4)=\sqrt{2}=1.41\cdots$, $D(5)=\frac{1}{2}(1+\sqrt{5})=1.61\cdots$, $D(6)=2\sin 72°=1.90\cdots$. Further, Theorem 3 gives $D(7)=2$. For $n=3$, 4, 5 the vertices of the regular $n$-gon of side-length 1 give the minimum diameter; for $n=6$ the best result is given by the set of points consisting of the vertices and circumcenter of a regular pentagon of circumradius 1.

**2. Proof that Theorem 1 is equivalent to Theorem 2.** To show that Theorem 1 implies Theorem 2 we proceed as follows. Suppose we have $k$ points in a circle of radius 2 such that one of the points is at the center and all the mutual distances are at least 1. If we construct a circle of unit radius about each point and then apply Theorem 1 to this set of $k$ circles, with $U$ as the circle around the central point, we get $k\leqq 19$.

In showing that Theorem 2 implies Theorem 1 (and, in fact, in proving Theorem 2) we use a polar coordinate system with radial distance $\rho$ and amplitude $\theta$. The set of points such that $\rho\leqq t$ we denote by $C(t)$. Now we may assume that the $U$ of Theorem 1 is the unit circle of our polar coordinate system. Thus we have a set $\Delta$ of $k-1$ circles of radius at least 1 each one of which meets the the unit circle $U$, the center of no one of these $k$ circles (including $U$) lying in the interior of another. Then it suffices to show that we can construct a set $\Delta^*$ of $k-1$ points in $C(2)$ whose mutual distances and distances from the origin are all at least 1. We do this by choosing a point for $\Delta^*$ corresponding to each circle $D$ of $\Delta$ in the following way: if the center of $D$ lies in $C(2)$, we pick the center; if the center $X$ of $D$ lies outside of $C(2)$, we pick the point $R$ lying on the circle $\rho=2$ and having the same amplitude as $X$. As a result of this correspondence the circle of radius 1 about a point $Q$ of $\Delta^*$ is contained in the corresponding circle of $\Delta$; hence $Q$ is at distance at least 1 from the points of $\Delta^*$ which were originally centers of circles of $\Delta$. Thus it remains only to prove that if two circles of $\Delta$ have centers $X$ and $Y$ both of which lie outside of $C(2)$, then the corresponding points $R$ and $S$ of $\Delta^*$ have mutual distance at least 1. Let $OX=x$, $OY=y$, angle $XOY=\psi$. Then by the properties of $\Delta$ we have $\overline{XY}^2\geqq\max\{(x-1)^2,(y-1)^2\}$; that is,

$$x^2+y^2-2xy\cos\psi\geqq\max\{(x-1)^2,(y-1)^2\}.$$

Now if we suppose that $y\geqq x$ we have

$$\cos\psi \leqq \frac{x^2+y^2-(y-1)^2}{2xy} = \frac{1}{x}+\frac{x^2-1}{2xy}$$
$$\leqq \frac{1}{x}+\frac{x^2-1}{2x^2} = 1-\frac{1}{2}\left(1-\frac{1}{x}\right)^2 \leqq \frac{7}{8},$$

since $x>2$. Therefore $\overline{RS}^2=8-8\cos\psi\geqq 1$. Thus the points chosen for $\Delta^*$ have the desired properties and our proof of equivalence is finished.

**3. Proof of Theorem 2.** The proof is based upon the following lemmas, of which the first is basic and the others are simple corollaries thereof.

LEMMA 1. *Let $r$ and $R$ be such that $0<R-1\leqq r\leqq R$ and suppose that we have two points $P$ and $Q$ which lie in the annulus $r\leqq\rho\leqq R$ and which have mutual distance at least* 1. *Then the minimum $\phi(r, R)$ of the angle $POQ$ has the following values*

$$\phi(r, R) = \arccos\frac{R^2+r^2-1}{2Rr}, \qquad \text{if } R-1\leqq r\leqq R-\frac{1}{R};$$

$$\phi(r, R) = \arccos\left(1-\frac{1}{2R^2}\right) = 2\arcsin\frac{1}{2R}, \qquad \text{if } R-\frac{1}{R}\leqq r\leqq R.$$

*In particular*

| | |
|---|---|
| $\phi(1, 1.10) > 54.^\circ0$ | $\phi(1.10, 2) > 16.^\circ8$ |
| $\phi(1, 1.15) > 51.^\circ5$ | $\phi(1.15, 2) > 20.^\circ0$ |
| $\phi(1, 1.20) > 49.^\circ2$ | $\phi(1.20, 2) > 22.^\circ3$ |
| $\phi(1, 1.25) > 47.^\circ1$ | $\phi(1.25, 2) > 24.^\circ1$ |
| $\phi(1, 1.30) > 45.^\circ2$ | $\phi(1.30, 2) > 25.^\circ5$ |
| $\phi(1, 1.45) > 40.^\circ3$ | $\phi(1.45, 2) > 28.^\circ3$ |
| $\phi(1, 1.60) > 36.^\circ4$ | $\phi(1.60, 2) > 28.^\circ9$ |

*Proof.* It suffices to consider the case in which $OQ=R$ and $PQ=1$. If we put $OP=\rho$, then our problem is to find the minimum of $f(\rho)=$angle $POQ$ $=\arccos\{(R^2+\rho^2-1)/(2R\rho)\}$ for $\rho$ in the interval $r\leqq\rho\leqq R$. By differential calculus we see that although $f(\rho)$ may have an interior maximum for $\rho=(R^2-1)^{1/2}$, it cannot have an interior minimum. If we compare $f(r)=\arccos\{(R^2+r^2-1)/(2Rr)\}$ and $f(R)=\arccos\{1-1/(2R^2)\}$, we see that when $R-1\leqq r\leqq R-1/R$ the minimum is achieved for $OQ=R$, $PQ=1$, and $OP=r$, while when $R-1/R\leqq r\leqq R$ the minimum is achieved for $OQ=R$, $PQ=1$, and $OP=R$.

We shall find it convenient in what follows to use the term admissible points to refer to a set of points in $1\leqq\rho\leqq 2$ whose mutual distances are all at least 1.

LEMMA 2. *There are at most* 12 *admissible points in the annulus* $1.45 \leqq \rho \leqq 2$.

*Proof.* This follows from the fact that $13\phi(1.45, 2) > 360°$. (Note that the constant 1.45 could be replaced by any number $r$ such that $\phi(r, 2) > 360°/13$, for example by 1.402.)

LEMMA 3. *It is impossible to have* 7 *admissible points in* $C(1.15)$, 8 *admissible points in* $C(1.30)$, 9 *admissible points in* $C(1.45)$, *or* 10 *admissible points in* $C(1.60)$.

*Proof.* In fact, $7\phi(1, 1.15) > 360°$, $8\phi(1, 1.30) > 360°$, $9\phi(1, 1.45) > 360°$, $10\phi(1, 1.60) > 360°$. (Note that the constants 1.15, 1.30, 1.45, 1.60 could be replaced by any values less than $\frac{1}{2}$ cosec $(180°/7)$, $\frac{1}{2}$ cosec $22°.5$, $\frac{1}{2}$ cosec $20°$, $\frac{1}{2}$ cosec $18°$, respectively. This remark, along with the fact that the vertices of the regular $k$-gon of side length 1 with center at the origin do constitute a set of $k$ admissible points in $C\{\frac{1}{2}$ cosec $(180°/k)\}$, provides the evaluation of $r(8)$, $r(9)$, $r(10)$, $r(11)$).

LEMMA 4. *It is impossible to have* 7 *admissible points in* $C(1.30)$ *such that* 6 *of them lie in* $C(1.10)$.

*Proof.* If we had 7 admissible points in $C(1.30)$, 6 of which lay in $C(1.10)$, then of the 7 angles subtended at $O$ by pairs of consecutive points (considered in order of amplitude) 5 would be each at least $\phi(1, 1.10)$ and the other two would be each at least $\phi(1, 1.30)$. But $5\phi(1, 1.10) + 2\phi(1, 1.30) > 360°.4$.

LEMMA 5. *It is impossible to have* 8 *admissible points in* $C(1.45)$ *such that* 7 *of them lie in* $C(1.25)$. *It is impossible to have* 8 *admissible points in* $C(1.45)$ *such that* 6 *of them lie in* $C(1.15)$.

*Proof.* The proof is similar to that of Lemma 4. The first part follows from the fact that $6\phi(1, 1.25) + 2\phi(1, 1.45) > 363°.2$; the second from the fact that $4\phi(1, 1.15) + 4\phi(1, 1.45) > 367°.2$.

Now we come to the proof of Theorem 2. We suppose that we have 19 admissible points and seek to get a contradiction. By Lemma 2 there are at most 12 admissible points outside $C(1.45)$ and by Lemma 3 at most 8 admissible points in $C(1.45)$. Thus there are two cases to consider, according to whether we have 7 admissible points in $C(1.45)$ and 12 admissible points outside, or 8 admissible points in $C(1.45)$ and 11 admissible points outside. The first case we subdivide further, according to whether one of the 7 points lies outside of $C(1.30)$ or not.

*Case* Ia: 12 admissible points $B_1, \cdots, B_{12}$ outside of $C(1.45)$, 7 admissible points $A_1, \cdots, A_7$ in $C(1.45)$, one of the $A_i$, say $A_k$, outside of $C(1.30)$. All the angles $B_iOB_{i+1}$ exceed $\phi(1.45, 2) > 28°.3$. (We mean to include the angle $B_{12}OB_1$, of course; similarly in the sequel.) But for any $i$ the angle $A_kOB_i$ exceeds $\phi(1.30, 2) > 25°.5$ and so one of the angles $B_iOB_{i+1}$ exceeds $2\phi(1.30, 2) > 51°.0$. However $11\ (28°.3) + 51°.0 = 362°.3$, a contradiction.

*Case* Ib: 12 admissible points $B_1, \cdots, B_{12}$ outside of $C(1.45)$, 7 admissible points $A_1, \cdots, A_7$ in $C(1.30)$. By Lemma 3 one of the $A_i$, say $A_k$, lies outside of $C(1.15)$ and by Lemma 4 another one of the $A_i$, say $A_m$, lies outside of $C(1.10)$. For each $i$ the angle $A_kOB_i$ exceeds $\phi(1.15, 2) > 20°.0$ and the angle $A_mOB_i$ exceeds $\phi(1.10, 2) > 16°.8$. Hence an angular sector of more than $40°.0$ and another sector of more than $33°.6$ are ruled out as possible locations for points $B_i$. These sectors cannot overlap, since the angle $A_kOA_m$ is at least $\phi(1, 1.30) > 45°.2$. If one of the angles $B_iOB_{i+1}$, say $B_nOB_{n+1}$, includes both these sectors, then $B_nOB_{n+1}$ exceeds $40°.0 + 33°.6 = 73°.6$ and we have a contradiction, since $11\phi(1.45, 2) + 73°.6 > 384°.9$. Otherwise we see that out of the 12 angles $B_iOB_{i+1}$ one exceeds $40°.0$ and another exceeds $33°.6$. Since at most 9 admissible points can lie in $C(1.60)$, at least 10 of the 12 points $B_i$ lie outside $C(1.60)$. Hence at least 8 of the 12 angles $B_iOB_{i+1}$ exceed $\phi(1.60, 2) > 28°.9$. But $40°.0 + 33°.6 + 6(28°.9) + 4\phi(1.45, 2) > 360°.2$, a contradiction.

*Case* II: 11 admissible points $B_1, \cdots, B_{11}$ outside $C(1.45)$, 8 admissible point $A_1, \cdots, A_8$ in $C(1.45)$. By Lemma 3 one of the $A_i$, say $A_k$ lies outside $C(1.30)$; by Lemma 5 a second one of the $A_i$, say $A_m$, lies outside of $C(1.25)$ and a third one of the $A_i$, say $A_n$, lies outside of $C(1.15)$.

Now for each $i$ the angle $B_iOA_k$ exceeds $\phi(1.30, 2) > 25°.5$, the angle $B_iOA_m$ exceeds $\phi(1.25, 2) > 24°.1$, and the angle $B_iOA_n$ exceeds $\phi(1.15, 2) > 20°.0$. Hence three angular sectors of more than $51°.0$. $48°.2$, and $40°.0$, respectively, are ruled out as possible locations for the points $B_i$. If one of the angles $B_iOB_{i+1}$, say $B_pOB_{p+1}$, includes two of these sectors, then $B_pOB_{p+1}$ exceeds $20°.0 + 24°.1 + 40°.3 = 84°.4$, since $A_kOA_m$, $A_mOA_n$, and $A_nOA_k$ are each at least $\phi(1, 1.45) > 40°.3$; this gives a contradiction, since $10\phi(1.45, 2) + 84°.4 > 367°.4$.

On the other hand if no angle $B_iOB_{i+1}$ includes two of the proscribed sectors, then we see that out of the 12 angles $B_iOB_{i+1}$ one exceeds $51°.0$, another exceeds $48°.2$, another exceeds $40°.0$, and each of the 8 remaining exceeds $\phi(1.45, 2) > 28°.3$. But $51°.0 + 48°.2 + 40°.0 + 8(28°.3) = 365°.6$, a contradiction.

**4. Proof of Theorem 3.** We require the following lemmas, of which the first three are well-known.

LEMMA 6. *The area of a triangle does not exceed $\frac{1}{4}\sqrt{3}$ times the square of the longest side.*

LEMMA 7. *The product of the diagonals of a quadrilateral is at least twice the area.*

LEMMA 8. *If a convex polygon has perimeter not less than $2\pi$, its diameter exceeds 2.*

LEMMA 9. *If there is a point in a triangle whose distance from each vertex is at least 1, then some side of the triangle has length at least $\sqrt{3}$.*

*Proof.* One of the sides of the triangle must subtend an angle of 120° or more at the point in question.

LEMMA 10. *If* $0<\xi\leqq\frac{1}{3}\pi$ *and* $ABCD$ *is a convex quadrilateral with* $\sphericalangle ABC=\frac{2}{3}\pi+\xi$, $\sphericalangle BCD\geqq\frac{2}{3}\pi-\frac{2}{3}\xi$, *and* $1\leqq AB, BC, CD\leqq 2$, *then* $AD>2$.

*Proof.* Put $AB=x$, $BC=y$, $CD=z$, $\sphericalangle BCD=\theta$. Then

$$\begin{aligned}\overline{AD}^2 &= \{y - x\cos(\tfrac{2}{3}\pi+\xi) - z\cos\theta\}^2 + \{x\sin(\tfrac{2}{3}\pi+\xi) - z\sin\theta\}^2\\ &= x^2+y^2+z^2-2yz\cos\theta+2zx\cos(\theta+\tfrac{2}{3}\pi+\xi)-2xy\cos(\tfrac{2}{3}\pi+\xi).\end{aligned}$$

Considering this expression as a function of $x, y, z, \theta$ over the domain $1\leqq x, y, z\leqq 2$, $\frac{2}{3}\pi-\frac{2}{3}\xi\leqq\theta\leqq\pi$, we see from the positivity of the partial derivatives that the smallest value of $\overline{AD}^2$ occurs for $x=y=z=1$, $\theta=\frac{2}{3}\pi-\frac{2}{3}\xi$. It suffices therefore to prove that

$$f(\xi) = 3 - 2\cos(\tfrac{2}{3}\pi-\tfrac{2}{3}\xi) + 2\cos(\tfrac{4}{3}\pi+\tfrac{1}{3}\xi) - 2\cos(\tfrac{2}{3}\pi+\xi)$$

exceeds 4 for $0<\xi\leqq\frac{1}{3}\pi$; but this follows from the fact that $f(0)=4$, $f(\frac{1}{3}\pi)=5-4\cos(4\pi/9)$, and $f''(\xi)<0$ for $0\leqq\xi\leqq\frac{1}{3}\pi$.

LEMMA 11. *If* $ABCDE$ *is a convex pentagon such that* $AB, BC, CD, DE\geqq 1$, $AE\geqq\sqrt{3}$, *and* $\sphericalangle C>120°$, *then either the diagonal* $AD>2$ *or the diagonal* $BE>2$.

*Proof.* If $\sphericalangle A>90°$, then $BE>2$; if $\sphericalangle E>90°$, then $AD>2$. Suppose then that neither angle adjacent to $AE$ exceeds 90°. Then if $\sphericalangle C=120°+\xi$, we see that either $\sphericalangle B\geqq 120°-\frac{1}{2}\xi$ or $\sphericalangle D\geqq 120°-\frac{1}{2}\xi$, for otherwise the angle sum of the pentagon would be less than $90°+90°+(120°+\xi)+2(120°-\frac{1}{2}\xi)=540°$. Hence by Lemma 10 either $AD>2$ or $BE>2$.

Now we turn to the proof of Theorem 3. Suppose then that we have seven points in the plane such that the mutual distances are all at least 1. In proving that the diameter of the set of seven points is at least 2, we consider five cases, according as the convex hull of the seven points is a triangle, quadrilateral, pentagon, hexagon, or heptagon. As stated in the theorem we shall find that the diameter actually exceeds 2 throughout, except for the subcase of the hexagonal case in which the angles of the hexagon are all 120°, the sides all have length 1, and the seventh point is at the center.

*The triangular case.* Suppose that a circle of radius $\frac{1}{2}$ is drawn about each of the seven points as center. Then no two of these circles can properly intersect. If one side of the triangle is intersected by the circles around two of the four inner points, then that side has length at least $3\sqrt{3}/2$ by the Pythagorean theorem. If no side of the triangle is intersected by two of the circles about inner points, then the area of the triangle is greater than the area of three circles of radius $\frac{1}{2}$, namely $3\pi/4>\sqrt{3}$; thus by Lemma 6 at least one side of the triangle has length greater than 2.

*The quadrilateral case.* Again we construct a circle of radius $\frac{1}{2}$ about each of the seven points. If one side of the quadrilateral is intersected by the circles about two of the three inner points, then that side has length at least $3\sqrt{3}/2>2$. Suppose then that no side is intersected by more than one of the circles about

the inner points. If just one or two sides are intersected by inner circles, then the area of the quadrilateral is greater than the area of three circles of radius $\frac{1}{2}$, namely $3\pi/4>2$, and thus by Lemma 7 at least one of the diagonals of the quadrilateral has length greater than 2.

Now notice that if a side of the quadrilateral is intersected by the circle around one of the inner points, then that side has length at least $\sqrt{3}$ by the Pythagorean theorem. If three sides are intersected by inner circles and the quadrilateral is a rectangle, then all sides of the rectangle have length at least $\sqrt{3}$ and hence the diagonals have length at least $\sqrt{6}$. If three sides are intersected by the circles about inner points and one of the angles of the quadrilateral exceeds 90°, then at least one of the sides adjoining that angle has length $\sqrt{3}$ or greater, while the other side adjoining it has length at least 1; hence (by the law of cosines) the diagonal spanning this angle has length greater than 2.

*The pentagonal case.* Again if the circles of radius $\frac{1}{2}$ about the two inner points intersect the same side of the pentagon, then that side has length at least $3\sqrt{3}/2$. On the other hand if these two circles intersect two different sides of the pentagon, then both these sides have lengths at least $\sqrt{3}$, the pentagon has perimeter at least $3+2\sqrt{3}>2\pi$, and thus by Lemma 8 the diameter of the pentagon exceeds 2.

Thus we assume that at least one of the inner points, say $K$, has a distance from the perimeter of the pentagon greater than $\frac{1}{2}$. Then the other inner point, say $L$, must lie in one of the triangles formed by $K$ and consecutive vertices of the pentagon, say the triangle $KAB$. By Lemma 9 one of the sides of the triangle $KAB$ has length at least $\sqrt{3}$. If either $AK$ or $BK$ has length $\sqrt{3}$ or more, the diameter of the pentagon exceeds $\sqrt{3}+\frac{1}{2}>2$. We suppose then that $AB \geqq \sqrt{3}$. Since one of the angles $ALK$ and $BLK$ is at least 90°, either $AK \geqq \sqrt{2}$ or $BK \geqq \sqrt{2}$. Suppose $AK \geqq \sqrt{2}$. Then prolong $AK$ until it meets the pentagon again at a point $X$. If $AX>2$, we are finished. If $AX \leqq 2$, then $KX \leqq 2-\sqrt{2}$, the side on which $X$ lies has length at least $2\{1-(2-\sqrt{2})^2\}^{1/2}>1.62$, the pentagon has perimeter greater than $3+1.73+1.62=6.35>2\pi$, and thus the pentagon has diameter more than 2 by Lemma 8.

*The heptagonal case.* This case is settled immediately by Lemma 8. Actually by pushing our methods further it is possible to prove that of all convex heptagons with sides of length at least 1 the minimum diameter occurs for the regular heptagon of side-length 1, in which case it is $1+2\cos(2\pi/7)=2.24\cdots$.

*The hexagonal case.* Dismissing the case in which a side of the hexagon has length greater than 2, we separate the proof into cases according to the number and arrangement of those angles of the hexagon which exceed 120°. First of all we note that if two adjacent angles of the hexagon exceed 120°, the diagonal spanning these two angles has length greater than 2. This case occurs, for example, if there are four or five angles of the hexagon exceeding 120°.

If exactly one angle of the hexagon exceeds 120°, say is $120°+\xi$, then one of the angles adjacent to it is at least $120°-\frac{1}{2}\xi$; the proof is completed in this case by Lemma 10. If exactly two angles of the hexagon exceed 120°, say have the

values $120°+\alpha$ and $120°+\beta$, and these two angles are non-adjacent, then somewhere in the hexagon we have a pair of adjacent angles of which one is $120°+\xi$, $\xi>0$, and the other is at least $120°-\frac{2}{3}\xi$; for otherwise the pentagon would have an angle sum less than $(120°+\alpha)+(120°+\beta)+(120°-\frac{2}{3}\alpha)+(120°-\frac{2}{3}\beta)+(120°-\frac{2}{3}\max\{\alpha, \beta\})+120°\leqq 720°$. Again the proof is completed by Lemma 10.

In case that exactly three of the angles of the hexagon exceed 120° and no two of these are consecutive, let $A, B, C, D, E, F$ be the vertices of the hexagon and let $A, C, E$ be those at which the angles exceed 120°. First we remark that if one of the sides of the hexagon has length greater than $\frac{1}{2}(\sqrt{13}-1)$, we have a diagonal of length greater than 2 by the law of cosines; thus we may assume all sides of the hexagon to have length not exceeding $\frac{1}{2}(\sqrt{13}-1)<\sqrt{3}$. The three diagonals $AC$, $CE$, $EA$ divide the hexagon up into four triangles. One of these four triangles contains the seventh point and hence one of the three diagonals $AC$, $CE$, $EA$ has length at least $\sqrt{3}$ by Lemma 9. Suppose $EA\geqq\sqrt{3}$. Then by applying Lemma 11 to the pentagon $ABCDE$ we find a diagonal of length greater than 2.

The only case left is that in which no angle of the hexagon exceeds 120°, *i.e.*, all angles are 120°. We easily see that every diagonal spanning two angles has length at least 2 and that the diameter is exactly 2 if and only if all the sides of the hexagon have length 1 (with the seventh point at the center naturally).

**5. Further questions.** Another function which we could consider is the diameter $d(n)$ of the smallest circle containing $n$ points whose mutual distances are all at least 1, without the restriction that one point be at the center. Obviously $D(n)\leqq d(n)\leqq 2r(n)$ and the same general remarks that were made about $D(n)$ and $r(n)$ could be made about $d(n)$. The first few values of $d(n)$ are $d(2)=1$, $d(3)=2/\sqrt{3}=1.15\cdots$, $d(4)=\sqrt{2}=1.41\cdots$, $d(5)=\operatorname{cosec} 36°=1.70\cdots$, $d(6)=2$, $d(7)=2$.

Naturally we can consider the analogue of $d(n)$ for other figures than the circle, for example, the side-length $t(n)$ of the smallest equilateral triangle containing $n$ points whose mutual distances are all at least 1. An interesting unsolved question about $t(n)$ is whether or not $t(\frac{1}{2}k\{k+1\}+1)>k-1$ for $k$ a positive integer. Obviously $t(\frac{1}{2}k\{k+1\})\leqq k-1$, since the regular hexagonal lattice gives $\frac{1}{2}k\{k+1\}$ admissible points in a triangle of side-length $k-1$.

In the proof that $D(7)=2$ we remarked that of all convex heptagons whose sides have length 1 or more the minimum diameter is assumed by the regular heptagon of side-length 1. An analogous statement can be made for triangles, quadrilaterals, and pentagons. However for hexagons the minimum diameter is assumed by the equilateral hexagon of side-length 1 whose angles are alternately 90° and 150°. The situation for $n$-gons with $n>7$ is an open question.

Another interesting problem is whether or not $r(n+1)>r(n)$ for $n\geqq 7$ and whether $D(n+1)>D(n)$ for $n\geqq 3$. It follows from Theorem 3 that $D(8)>D(7)$

$=2$, since the diameter of a set of seven points (with mutual distances 1 or more) is greater than 2 except for one special configuration (for which we can make a separate argument).

Of course we can consider the analogues of $r(n)$ and $D(n)$ in $k$ dimensions, say $r_k(n)$ and $D_k(n)$. Clearly $D_k(n)=1$ for $n \leqq k+1$. It would be interesting to have a good estimate for $D_k(k+2)$ from below. It is probably true that $D_k(k+2) \geqq \sqrt{4/3}$, but this seems difficult to prove. On the other side it is easy to give an example to show that $D_k(k+2) \leqq \sqrt{4k/(3k-2)}$. As for $r_k(n)$, we do not even know the smallest value of $n$ such that $r_k(n)>1$, except for $k=2$. For $n \to \infty$ there are asymptotic relations of the form (cf. [2])

$$\tfrac{1}{2}D_k(n) \sim r_k(n) \sim c_k n^{1/k},$$

but $c_k$ is not known for $k>2$.

**6. Added in Proof:** After this paper was submitted for publication, a proof of our Theorems 1 and 2 was published by E. F. Reifenberg, Math. Gaz., vol. 32, 1948, pp. 290–292. His proof is in general similar to ours, although considerably different in detail.

The problem mentioned briefly in the third paragraph of §5 has been discussed recently in a paper by S. Vincze, Acta Univ. Szeged. (Sect. Sci. Math.) vol. 12, part A, 1950, pp. 136–142. For $n$ not a power of 2 he shows that the minimum diameter possible for convex $n$-gons whose sides have length at least 1 is $\frac{1}{2}\operatorname{cosec}(\pi/2n)$. In particular for even $n$ having at least one odd prime factor the regular $n$-gon of side length 1 does not give the minimal diameter. Vincze points out that a closely related problem was considered earlier by K. Reinhardt, Jber. Deutsch. Math. Verein., vol. 31, 1922, pp. 251–270.

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# AREA IN NON-EUCLIDEAN GEOMETRY

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**1. Introduction.** Because of its origin in connection with the problem of parallelism, hyperbolic geometry, the non-Euclidean geometry of Lobatchewsky and Bolyai, has from the beginning been treated in as close analogy as possible with Euclidean; particularly in synthetic discussions. However, it has long been known that hyperbolic geometry is perfectly realized on a sphere of imaginary radius. The writer believes that the key to a quick intuitive grasp of this geometry is to minimize the analogy with the Euclidean plane, and to stress instead the analogy with the real sphere.

On the sphere (or in the "elliptic plane," which may be regarded as the sphere with opposite points abstractly identified) any two "lines" both intersect and have a unique common perpendicular. In the hyperbolic plane line pairs either intersect or have a unique common perpendicular, "parallels" being the singular, intermediate case. On the sphere the common normal of two lines is greater than any other distance to one from a point of the other, while in the hyperbolic plane the common normal of two "ultra-parallels" is the shortest distance between them, and away from it in either direction the lines diverge indefinitely. Parallels are asymptotic.

The angular excess of a spherical triangle is replaced, in the hyperbolic plane, by angular deficiency. The theorem that the defect of the sum of two polygons is the sum of their defects is proved precisely as in the case of the excess. As regards area, however, there is an important difference, for the hyperbolic plane is infinite in extent. This means that the fundamental theorem, that area is proportional to deficiency, must be handled differently.

The customary treatment is in part remarkably like that of Euclid, an essential procedure being to dissect the triangle in such a way as to show it equivalent to a quadrilateral with two right angles. This achieves the object neatly enough, however, and particularly as handled by Liebmann [1] seems entirely satisfactory.* But the same argument as treated by Carslaw [2] and others [3], and outlined by Wolfe in his recent textbook [4], seems to the writer less acceptable. These writers introduce the equivalence relation used by Hilbert [5] in his *Foundations of Geometry* to relate area to congruence. This relation makes two polygons equivalent if they can be partitioned into the same finite number of triangles, congruent in pairs. For Hilbert this relation serves the useful purpose of economizing on axioms, but by itself it is not an altogether suitable basis for a theory of area, for the area concept does not inherently require such a strong equivalence. The fact that polygons of equal area can be thus finitely partitioned has no more *a priori* force than its analogue in three dimensions for equal polyhedral volumes, which, as is well known, does not hold.

The present paper employs a more direct axiomatic approach. It also puts to use the analogy with the elliptic case, this being our chief motivation. There

* However this argument requires the use of equidistant curves.

is a further point of interest in the use that is made of a class of polygons most characteristic of the non-Euclidean planes, but frequently neglected, the orthogons—polygons all of whose angles are right. In the elliptic plane ortho-$n$-gons exist for $n=2$ and $n=3$; in the hyperbolic plane for $n\geqq 5$. The hyperbolic plane can be tessellated with duplicates of an arbitrary ortho-$n$-gon, $n\geqq 5$, without the requirement of regularity.

What makes the area problem so simple on the sphere is the finiteness of the total area and the existence of a triangle one angle of which is proportional to its area, the triangle with two right angles. The figure corresponding to this in the hyperbolic plane is not a triangle but a quadrilateral with three right angles. The area is not proportional to the remaining acute angle, but is proportional to its complement. This is the key theorem in the argument which follows; the fundamental theorem for the triangle is a simple consequence.

**2. The area function.** By the term "area" we shall mean here any function $\mu$ satisfying the following postulates:

1. The function $\mu$ is defined for every simple closed polygon.
2. The function $\mu$ is a real number, positive and finite.
3. If the simple closed polygons $s$ and $t$ are congruent, then $\mu(s)=\mu(t)$.
4. If the simple closed polygon $w$ is the sum of two others, $s$ and $t$, then $\mu(w)=\mu(s)+\mu(t)$.

For such a function we shall use the conventional terminology. We shall say that a polygon has "an area," one polygon has "greater area" than another, *etc.*

In the Euclidean plane there exists no intrinsic function $\mu$. Euclid, in effect, assumed the existence of such functions and then showed that any two can differ only by a factor of proportionality. In the non-Euclidean geometries the situation is different: angular excess in the elliptic plane, and defect in the hyperbolic, satisfy these postulates—the argument being familiar. What remains to be shown, in each case, is that *any* function satisfying them is proportional to the excess or defect. We are concerned with the hyperbolic plane only. To be completely explicit: writing $d(s)$ for the angular defect of $s$, we have to show that for any function $\mu$, we can write $\mu(s)=k^2d(s)$. Besides the existence of the angular defect, we shall use the axiom of Archimedes, and the basic theorems on lines, such as the uniqueness of the common normal of two ultra-parallels, and the theorem that away from their common normal ultra-parallels diverge.

**3. The area theorems.**

THEOREM 1. *If two trirectangles have equal acute angles, then they have equal area.*

Proof: Let the two trirectangles be $t=ABCD$ and $t'=KLMN$, the equal acute angles being $A$ and $K$, and let us suppose $\mu(t)\geqq\mu(t')$.

Let the figures be superimposed so that $KL$ falls along $AB$ and $KN$ along $AD$. (See Figure 1.) If $L$ now coincides with $B$, then $M$ must coincide with $C$,

for $CD$ is the unique common normal of the ultra-parallels $AD$ and $BC$. The trirectangles are thus congruent and have the same area by Postulate 3. If $L$ falls between $A$ and $B$, then $D$ will lie between $A$ and $N$. For, the only alternative is that $N$ lie between $A$ and $D$; but then $M$ would lie within $ABCD$, and the hexagon $LBCDNM$ would have no angular defect.

Let us suppose that $L$ does fall between $A$ and $B$, and let $LB$ and $CD$ intersect in $E$. We see that in this case $t$ and $t'$ are simultaneously decomposed into new quadrilaterals: a birectangular part in which they overlap, and new trirectangles, $BE$ and $EN$, which have equal acute angles at $E$.

These new trirectangles can be superimposed in the same way and the procedure can be carried on indefinitely, unless brought to a stop after a finite number of steps by the coincidence of the last pair of trirectangles. If it is thus brought to a stop we have $\mu(t)=\mu(t')$ by Postulates 3 and 4.

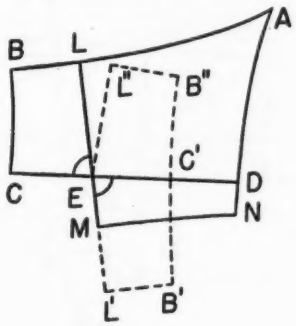


FIG. 1

To obtain a proof when the process is infinite, we must be more precise about the manner in which the successive superpositions are carried out. We shall arrange it in such a way that the process falls into distinct stages, each of which contains a finite number of superpositions. At the end of each stage it will be possible to show that the trirectangles which remain are each less than half the trirectangles with which that stage began.

Suppose in the initial superposition $EC<ED$ (Figure 1), and let the trirectangle $EB$ be rotated through a straight angle about $E$ to the position $EB'$. Let $EB'$ be reflected in $ED$, giving rise to the trirectangle $EL''B''C'$. This reflection lies wholly within $ALED$. Otherwise $EL''$ or $C'B''$ must have a point in common with $AL$, which is impossible. $EL''=EL$, and the perpendicular is shorter than any oblique segment from a point to a line. Also, by the divergence of ultraparallels, mentioned above, $C'B''=CB$ is shorter than any other perpendicular to $CD$ from a point of $AL$.

By Postulate 4, $EL''B''C'$ has a smaller area than $ALED$. And since, by assumption, $\mu(EB)\geqq\mu(EN)$, we have $\mu(BE)<1/2\cdot\mu(t)$ and $\mu(EN)<1/2\cdot\mu(t')$.

This is the goal of the first stage. We now let $BE=t_1$ and $EN=t_1'$, and proceed in the same way with the second stage.

Suppose, on the other hand, $EC>ED$ in the initial superposition. In this case, the first stage must contain further superpositions, as indicated in Figure 2. Upon rotation of $BE$ about $E$, the vertex $E_1$ of the new acute angles will fall on $AN$.

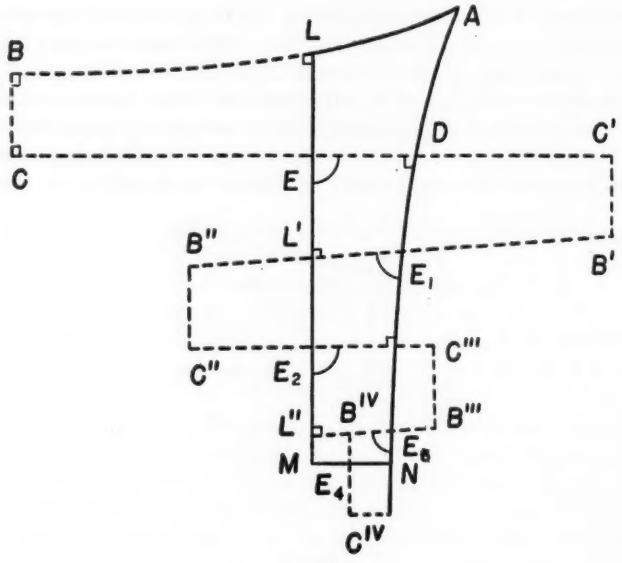


FIG. 2

We proceed to rotate $E_1C'$ about $E_1$, and compare $E_1B'$ with $E_1L'$. If $E_1B'>E_1L'$, as in the figure shown, we rotate again about $E_2$, *etc.* At each step in this process the sides of the successive trirectangles which are adjacent to $BC$, $B'C'$, *etc.*, are diminished by an amount greater than $MN$. It follows from the axiom of Archimedes that the sides will become in a finite number of steps less than $MN$. If we consider the various possible resulting configurations and argue from the angular defect as in the first paragraph of this proof, we can rule out all alternatives but that, for some $k$ we shall find $E_{k+1}$ on $MN$. By the same reasoning as before it will follow that $\mu(E_kC^k)<\mu(E_kE_{k-1})<\mu(E_kL)$. And since $\mu(E_kM)<\mu(E_kC^k)$ we have again reached our goal. We let $E_kC^k=t_1$, and $E_kM=t_1'$. After $n$ stages, we can write for the difference of the original areas:

$$\mu(t)-\mu(t')=\mu(t_n)-\mu(t_n')<\mu(t_n)<1/2\cdot\mu(t_{n-1})<1/2^n\cdot\mu(t).$$

The fixed numbers $\mu(t)$ and $\mu(t')$ therefore differ by as little as we please and must be equal.

For later reference, we state the following obvious corollary:

COROLLARY 1a. *In a variable trirectangle of fixed acute angle, either side adjacent to the acute angle can be arbitrarily increased without altering the area.*

(Note that the side in question cannot be arbitrarily diminished, since the adjacent sides must remain ultra-parallel.)

THEOREM 2. *If the acute angle of one trirectangle is greater than that of another, then its area is less.*

Proof: Let the trirectangle with the greater acute angle be $ABCD$, the acute angle being at $A$. Consider the side $A\bar{B}$ and one of the sides adjacent to the acute angle in the other trirectangle. Let them be made equal by extending the shorter one by Corollary 1a, and let these equal sides be superimposed as in Figure 3, where the second trirectangle is represented by $ABRS$. Segment $BR$

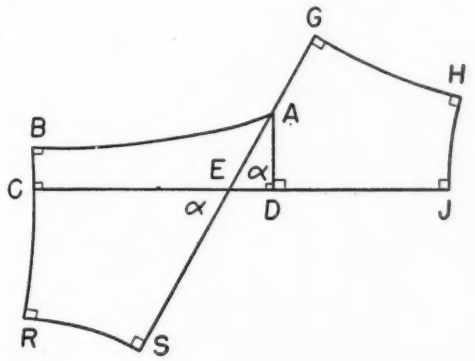


FIG. 3

will then be longer than $BC$. Let $AS$ meet $CD$ at $E$ and call the vertical acute angles $\alpha$. On $EA$ let segment $EG$ be marked off such that $EG > EA$ and also greater than the segment whose angle of parallelism is $\alpha$. Let $GH$ be perpendicular to $EG$. The line $GH$ is then ultra-parallel to $CD$; let the common normal be $HJ$. Segment $EJ$ is then definitely longer than $ED$. By Theorem 1 the trirectangles $ECRS$ and $EGHJ$ have the same area. Since triangle $EAD$ is a proper part of $EGHJ$, the theorem follows.

The following theorem is now evidently true:

THEOREM 3. *If two trirectangles have the same area they have the same acute angle.*

Thus the area of a trirectangle is a monotonic decreasing function of its acute angle, and a monotonic increasing function of its angular defect, which is just the complement of the acute angle. It remains to show that the latter function is a proportionality.

THEOREM 4. *All orthopentagons have the same area.*

Proof: If two orthopentagons have a side of one equal to a side of the other they can be superimposed, and the conclusion then follows by Theorem 1. But any side of an orthopentagon can be arbitrarily increased without changing the area; hence, the superposition can always be carried out.

Remark: For the area of an orthopentagon we shall use the letter $U$.

COROLLARY 4a. *The area of an ortho-n-gon is* $(n-4)U$.

Proof: If $s_1$, $s_2$, $s_3$, $s_4$ are four consecutive sides of any ortho-$n$-gon, then $s_1$ and $s_4$ are ultra-parallel. Their common normal lies inside the figure, and is the side opposite $s_2$ and $s_3$ of an orthopentagon. Removing from the figure this orthopentagon leaves an ortho-$(n-1)$-gon and diminishes the area by $U$.

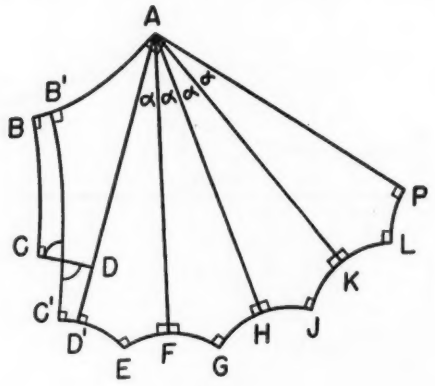


FIG. 4

THEOREM 5. *The area of a trirectangle is proportional to its angular defect.*

Proof: It is sufficient to show that if the ratio of the defects of two given trirectangles is $m/n$ then this is also the ratio of the areas.

Let the trirectangles be $t$ and $t'$, and let $d(t)/d(t') = m/n$. Let $t = ABCD$, with the acute angle at $A$. Let $AP$ be perpendicular to $AB$, as in Figure 4. The defect of $t$ is then angle $DAP$. Let that angle be divided into $m$ equal parts by the lines $AF$, $AH$, $AK$, *etc.*, the equal angles having the value $\alpha$. Now if side $AD$ is not greater than the segment whose angle of parallelism is $\alpha$, let it be extended by Corollary 1a, so that $C'D'$ is ultra-parallel to $AF$. Let the common normal be $EF$; $EF$ is then ultra-parallel to $AH$, by the equality of the angles $\alpha$, and similarly $GH$ to $AK$, *etc.* The resulting figure, represented in our figure by $AB'C'EGJLP$, is an ortho-$(m+4)$-gon whose area, by Corollary 4a, is $mU$.

If we write $\mu_\alpha$ for the area of one of the trirectangles with acute angle $\alpha$,

we have $\mu(t) = mU - m\mu_\alpha = m(U - \mu_\alpha)$. Now consider the trirectangle $t'$; let the complement of the acute angle be divided into $n$ equal parts—by hypothesis these equal angles are again $\alpha$. Proceeding as above, we obtain $\mu(t') = n(U - \mu_\alpha)$, which proves the theorem. For the trirectangle $t$ we may write $\mu(t) = k^2 d(t)$.

COROLLARY 5a. *The area unit $U$ can be written $k^2 \cdot \pi/2$.*

Proof: It is easily seen that an orthopentagon is the sum of two trirectangles whose total defect is $\pi/2$.

THEOREM 6. *The area of a triangle is proportional to its angular defect.*

Proof. Let $ABC$ be any (proper) triangle and let the sides be extended, $AB$ through $B$, $BC$ through $C$, and $CA$ through $A$. (See Figure 5). On these exten-

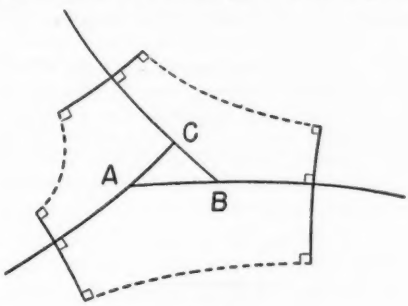


FIG. 5

sions let perpendiculars be erected, sufficiently far out to be pair-wise ultra-parallel. With the common normals, these then make an orthohexagon. Each vertex of the triangle is then the vertex of a pentagon with four right angles, a figure which is the sum of two trirectangles whose total defect is just the corresponding angle of the triangle. The area of the hexagon is $k^2\pi$. Subtracting the three pentagons, we have $\mu(\triangle ABC) = k^2(\pi - A - B - C)$.

**4. The factor of proportionality.** In the hyperbolic plane the factor of proportionality is strictly arbitrary; but it is natural, and customary, to make it agree with the otherwise locally Euclidean character of the plane. In a hyperbolic plane of parameter $\rho$ the factor is then $\rho^2$. This can be shown in various ways; the following argument is simpler than some.

Consider an isosceles trirectangle $t$, the length of the sides opposite the acute angle being $a$. Let the complement of the acute angle be $\theta$. By trigonometry we have then

$$\sin\theta = \sinh^2 \frac{a}{\rho}.$$

The area of $t$ is then

$$A = k^2 \arcsin\left(\sinh^2 \frac{a}{\rho}\right).$$

We want it to be true that

$$\lim_{a\to 0} A/a^2 = \lim_{a\to 0} 1/a^2 \cdot k^2 \operatorname{arc\,sin}\left(\sinh^2 \frac{a}{\rho}\right) = 1.$$

Since the sine and the hyperbolic sine both differ from their arguments by terms of the third degree and higher, we have at once $k^2 = \rho^2$.

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2. Carslaw, The Elements of Non-Euclidean Plane Geometry and Trigonometry, London, 1916, pp. 84–90.

3. See A. Finzel, Mathematische Annalen, Leipzig, 1912, pp. 262–284. Also, but with a difference: H. S. M. Coxeter, Non-Euclidean Geometry, Toronto, 1942, pp. 241–251. Professor Coxeter gives an interesting alternative to the usual treatment—it is something of a tour de force, beginning with the area of a horocyclic sector, a figure both curvilinear and unbounded.

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## FURTHER EXPERIENCE WITH UNDERGRADUATE MATHEMATICAL RESEARCH*

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At the 1941 summer meeting of the Association, I reported briefly on the experience of sixteen colleges and universities with undergraduate mathematical research, and described more fully the experience at Reed College with the senior thesis which is required for graduation. At that time there had been 94 theses in mathematics, of which approximately 80 were of a research type in the sense that they explored some new problem or a new aspect of an old problem and got some apparently new results. Details of my report can be found in this MONTHLY for June–July, 1942.†

Since 1941 we have had 41 mathematical theses, of which 34 have been of a research type. The average level of the research seems possibly a little higher than that reported on before. In the few minutes available I shall try to suggest the range of the problems considered, to characterize a few of the papers, and to

* Read before Sec. VII, International Congress of Mathematicians, Sept. 2, 1950.

† Vol. 49, No. 6, pages 379–385.

indicate how our department views the role of the thesis in undergraduate education.

Of thirteen theses in analysis, five were on problems in the calculus of variations. One of these gave a definitive treatment of the isoperimetric brachistochrone problem. Another dealt with constrained motion on a paraboloid, under gravity, exhibited possible paths by several systems of mapping, and determined the brachistochrone on the paraboloid. Three other theses investigated the geodesics on particular surfaces, one of which is an elliptical torus and another is the surface generated by revolving a lemniscate about its principal axis. Three theses dealt with the numerical or graphical solution of the differential equation $dy/dx=f(x, y)$. One of these set up very accurate formulas, of which Runge's and Kutta's formulas are special cases. Another, which generalized the type of formula used by Milne and others, got seven-place accuracy in the test case proposed by the National Research Council's committee on numerical integration, and without much labor. Three other theses dealt with special functions. One of these defined, analyzed, and applied hypocycloidal functions analogous to the circular and hyperbolic functions. Another, on functions with recurring derivatives, studied the solutions of $d^ny/dx^n=\pm y$, obtaining many identities and investigating the solutions of a related functional equation for the case $n=3$. (It was subsequently learned that about half of the results had been found previously by Glaisher and others.) Another thesis in analysis studied an infinite sequence generated by an exponential recurrence relation. Another, on the student's own initiative, constructed an extensive function theory in a hypercomplex number field.

Of twelve theses in geometry, four dealt with special surfaces and space curves. One of these obtained various properties of three-dimensional Lissajou curves, and from some of these properties derived much information about related plane curves. It also gave examples of the classification of non-intersecting, non-degenerate Lissajou space curves as topological knots by the calculation of torsion numbers. Another thesis in this group examined analytically some metrical properties of a two-parameter family of surfaces related to the Möbius strip, and suggested a method of classifying topologically this family and its intersections with some common surfaces. Another determined the complicated centro-surface for a special hypocycloidal surface. Still another geometrical thesis analyzed some alleged methods of trisecting an angle. Three others dealt with plane transformations and correspondences. One of these investigated the geometric properties of the analytic duals of several subgroups of the group of projective transformations, examining seven groups of affine transformations, and also showing a method of dualizing metric theorems. Another thesis defined pencils of curves of degree $n$ and order $m$, also the cross-ratio of four curves of such a pencil, and derived some properties of such cross-ratios. Still another constructed and studied a family of porisms relating to polygons circumscribed about a given polygon. (This thesis is part of a project now being carried on by one of the staff.)

Of two theses in algebra and number theory, one used the concept of the cycle of an integer $a$ (mod $m$) to develop the problem of power residues by what appears to be a new approach. It deduced some basic properties of the cycle and applied these to certain classes of integers.

In the field of foundations and abstract logic there have been four theses. One of these derived Hilbert's axioms, except that on parallels, from Pieri's postulates. Another was a particularly mature study in the algebra of logic. Among other contributions it constructed three systems of propositional calculi with multiple truth values, one a Boolean algebra and the others non-distributive lattices.

There have been ten theses in applied mathematics: two of these in actuarial science, two in mathematics of finance, four in theoretical or applied statistics and two in physical science. One thesis fitted Makeham's formula to a mortality table by least squares, and in solving the normal equations had to determine the location of the real roots of an equation of degree 266. Two others, supervised mainly by an economist, made valuable contributions to the mathematical study of statistical inference, examining the extent to which the $Z$ and $F$ tests of significance are applicable to non-normal populations and to binomial distributions. Still another thesis studied the effect of a cylindrical source of light within a parabolic reflector; and another made a highly technical analysis of sound-ranging in warfare, based on data from World War I.

You may be wondering what the seniors who wrote these and the earlier theses have done since graduation. A large majority of the men and many of the women have gone into mathematical, scientific, and actuarial pursuits. It recently came to our attention that the percentage of Reed male graduates since 1924 who have gone on to the doctorate in mathematics is the highest in the United States. Indeed, the absolute number of 1936–1945 mathematics doctors stemming from Reed is as large as from some of our noted universities. This may be due in part to the stimulus given by our thesis program, in part to the general intellectual atmosphere of the college, and perhaps in part to the interest aroused by our introductory work in mathematics which has led considerable numbers to select mathematics as their college major. Though a small college of about 600 students, we have 18 students registered this fall for an upper-division year-course in Theory of Equations and Modern Algebra.

Notwithstanding the types of career chosen by our graduates, we have never regarded the thesis project as training for graduate work or for a profession. Rather, it is the culmination of a program of general education with a special emphasis. The primary objective of all departments at Reed is to give students a broad perspective and a familiarity with fundamental problems, rather than to train them for a particular vocation. For some routine jobs specific training is, of course, indispensable; but for many vocations *education* is vastly more important than *training*; and a man with a sound grasp of basic theory and its significance will soon master the technical requirements of his particular job. And he will see beyond the job. For our seniors, the thesis topic, whether pro-

posed by the student himself or chosen by him from a wide range of suggestions, represents an expression of a scholarly interest. Advanced theoretical research usually arises out of intellectual curiosity rather than from the exigencies of a job; and the senior thesis project, on its more modest level, shares this quality.

While most of the chosen topics naturally relate to classical mathematics, there is opportunity, if the student so desires, to explore some phase of recent development. We realize that the content of undergraduate courses inevitably involves some lag behind recent discoveries, and we are glad when the thesis gives a student an opportunity for even a slight correction of that lag.

## MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, *University of California*

*Material for this department should be sent to F. A. Ficken, Institute of Mathematics and Mechanics, New York University, 45 Fourth Ave., New York 3, New York.*

### ARITHMETICO-GEOMETRICAL PROGRESSIONS HAVING THREE COMMON TERMS*

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**1. Introduction.** A series is called an arithmetico-geometrical progression (*AGP*) if each term of it is the product of corresponding terms of an arithmetic and a geometric progression [1]. It has been observed that different *AGP*'s may have three corresponding terms alike. For example, *AGP*'s having first, third, and fourth terms alike are

$$3+\frac{47}{5}+19-45-\cdots, \qquad R=5, \qquad D=-\frac{28}{25};$$

$$3-\frac{31}{4}+19-45+\cdots, \qquad R=-2, \quad D=\frac{7}{8};$$

$$3-\frac{23}{3}+19-45+\cdots, \qquad R=-3, \quad D=-\frac{4}{9};$$

here $R$ denotes the common ratio of the $GP$ component of the $AGP$, and $D$ the common difference of its $AP$ component.

This note continues previous work [2, 3] of the author and establishes a theorem on the number $AGP$'s having three given terms in common. We shall use the following lemma which is easily proved:

LEMMA. *If $a$, $b$, $c$ are respectively the* $f$th, $g$th, *and* $h$th *terms, $f<g<h$, of an AGP whose $R=t$, then $at^{h-f}$, $bt^{h-g}$, $c$ are respectively the* $f$th, $g$th, *and* $h$th *terms of a certain AP.*

* Revised by P. A. Clement.

**2. Number of such AGP.** We shall establish the following result.

THEOREM. *Let $l$, $m$, $n$ where $ln \neq 0$ be respectively the $p$th, $q$th, and $r$th terms, $p<q<r$, of an AGP. Then the number of distinct such AGP's is $r-p$ provided $m^{r-p} \neq n^{q-p}l^{r-p}$; if $m^{r-p}=n^{q-p}l^{r-p}$, the number of AGP's is $r-p-k$ where $k=(q-p, r-q)$, the greatest common divisor of $q-p$ and $r-q$.*

*Proof.* With $R$ denoting the common ratio of the $GP$ component of the $AGP$, it follows from the lemma that $lR^{r-p}$, $mR^{r-q}$, $n$ are respectively the $p$th, $q$th, and $r$th terms of a certain $AP$. Letting $A$ denote the first term of this $AP$ and $B$ its common difference, we have

$$lR^{r-p} = A + (p-1)B,$$
$$mR^{r-q} = A + (q-1)B,$$
$$n = A + (r-1)B.$$

When these equations are multiplied respectively by $q-r$, $r-p$, and $p-q$ and then added, we get

$$l(q-r)R^{r-p} + m(r-p)R^{r-q} + n(p-q) = 0. \tag{1}$$

It is clear that $l$, $m$, and $n$ are common to $AGP$'s whose $R$'s satisfy (1); hence these quantities are respectively the $p$th, $q$th, and $r$th terms of $AGP$'s whose $R$'s are the distinct roots of the equation

$$F(x) \equiv l(q-r)X^{r-p} + m(r-p)X^{r-q} + n(p-q) = 0. \tag{2}$$

If (2) has a double root, it must satisfy $F'(x)=0$ and hence, since $x \neq 0$, the equation

$$lx^{q-p} - m = 0. \tag{3}$$

Moreover, this same root must satisfy the equation $(r-p)F(x)-xF'(x)=0$, which yields

$$mx^{r-q} - n = 0. \tag{4}$$

Elimination of $x$ from (3) and (4) presents the condition

$$m^{r-p} - n^{q-p}l^{r-q} = 0. \tag{5}$$

Hence the number of distinct $AGP$'s is less than $r-p$ if (5) holds; otherwise it is exactly $r-p$.

Now assume that (5) holds and that $(q-p, r-q)=k$. Then an application of the Euclidean algorithm to the polynomial functions of (3) and (4) readily determines their highest common factor to be $x^k-b$, where $b$ is a constant. This factor gives all the double roots of (2), and thus (2) admits exactly $k$ double roots. Thus the number of distinct $AGP$'s for this case is $r-p-k$.

**3. Remark.** It may be verified that repeated roots occur only when the given

terms $l$, $m$, $n$ belong to a $GP$, and that then the $k$ roots of the equation $X^k - (m/l)^{k/q-p} = 0$ are all the double roots of (2).

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## POLYNOMIAL PARAMETRIZATIONS

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**1. Introduction.** If $P(t)$ is any polynomial in $t$ with coefficients 0 or 1, then the $2\times 2$ matrix

$$U = \begin{pmatrix} P, & 1+P \\ 1+P, & 1 \end{pmatrix}$$

is orthogonal, modulo 2; that is, $UU^T = 1 \pmod 2$. However, for prime modulus $p$, $p>2$, there can be no non-constant $2\times 2$ orthogonal matrices, the elements of which are modular polynomials. This fact follows from the theorem established in this note; the theorem and its corollaries seem to have interest in themselves. The truth of the theorem was conjectured by D. H. Lehmer in conversation.

**2. Theorem.** In the following, $R$ denotes a unique factorization domain (that is, a commutative ring with unit element in which the cancellation law holds and factorization is unique up to associates); $R[x, y, t]$ is the ring of polynomials in $x$, $y$, $t$ with coefficients in $R$; and $C_p$ is the finite field with $p$ elements, where $p$ is a prime.

THEOREM. *Let $f(x, y)$ be a homogeneous polynomial of degree $m$, $f(x, y) \in R[x, y]$; suppose that non-constant polynomials $P=P(t)$, $Q=Q(t)$ exist, $P$, $Q\in R[t]$, such that $G(t)=f(P, Q)$ is a non-zero constant in $R[t]$. Then $f(x, y) = c(bx-ya)^m$; that is, $f(x, y)$ is a constant times the mth power of a linear form.*

*Proof.* Assume that $m$ is positive. Let $f(x, y)$, $P$, $Q$, satisfy the hypotheses of the theorem, where

$$P(t) = \sum a_i t^i, \qquad Q(t) = \sum b_i t^i. \tag{1}$$

It is easy to see that degree $P=$ degree $Q=n$, say. It will turn out that $b_n=b$, $a_n=a$. The coefficient of $t^{mn}$ in $G(t)$ must be zero; therefore $f(a_n, b_n)=0$. Thus $f(x, y)$ is divisible by $b_nx-ya_n$.

If the quotient $g(x, y)$ in this division has positive degree, then the quotient must be divisible by $b_nx-ya_n$, since $g(P, Q)$ must be a non-zero constant. The theorem follows by induction.

**3. Corollaries.** We have the following consequent results.

COROLLARY 1. *If $f(x, y)$ is any homogeneous polynomial of degree $m$, and $h$ is any constant, $f(x, y)$, $h \in R[x, y, t]$, and if the equation $f(x, y) = h$ has polynomial parametrizations $x = P(t)$, $y = Q(t)$, then the same equation has parametrizations of degrees* 1, 1.

COROLLARY 2. *In Corollary* 1, *if $h \neq 0$, then all polynomial parametrizations, if there are any, are given by $x = aS(t) + cw$, $y = bS(t)$, where $a$, $b$, $c$ are constants whose values are fixed once $f(x, y)$ and $h$ are given; $S(t)$ is an arbitrary polynomial, and $w$ is an arbitrary $m$th root of unity.*

Corollary 2 gives a parametrization of the parametrizations.

COROLLARY 3. *The conclusions of the above theorem and corollaries are true in the more general case in which $t$ is a set of indeterminates rather than a single one.*

*Proof.* Let $f(x, y)$ be a homogeneous polynomial of positive degree; let $t$ stand for a set $t_1, t_2, \cdots, t_s$ of indeterminates; let $P(t)$, $Q(t)$ be polynomials in these indeterminates. Let $G(t) = f[P(t), Q(t)]$ be a constant in $R[x, y, t]$. If $P(t)$, $Q(t)$ did not have the same degree, the coefficient of the first* term in $G(t)$ could not be zero. Indeed the first term of $P(t)$ must be the same (except for the coefficient) as the first term of $Q(t)$. Let the coefficients of these first terms be $a$, $b$ respectively; let the terms themselves be called $w$, $v$. Then $f(w, v) = 0$, so that $f(a, b) = 0$, and $f(x, y)$ must be divisible by $bx - ya$, as before.

COROLLARY 4. *Let $f(x, y)$ be a homogeneous polynomial of discriminant $D$ with integral coefficients, not identically an integer times the $m$th power of a linear form. Then the congruence $f(x, y) \equiv K \pmod{p}$, $K \neq 0$, can have polynomial parameterizations for only a finite number of primes.*

This is the form in which Lehmer made his conjecture. Clearly $p \mid DK$ for only a finite number of primes.

COROLLARY 5. *The only $2 \times 2$ orthogonal matrices with elements in $C_p[t]$, $p > 2$, are the constant matrices. When $p = 2$, the only $2 \times 2$ orthogonal matrices are*

$$\begin{pmatrix} 1 + P & P \\ P & 1 + P \end{pmatrix},$$

*where $P(t)$ is an arbitrary polynomial.*

*Proof.* If $U = (u_{ij})$ is orthogonal, then $u_{11}^2 + u_{12}^2 = 1$. The polynomial $x^2 + y^2$ has discriminant 4, so $x^2 + y^2$ does not have multiple factors in $C_p[x, y]$, $p > 2$. Moreover, $x^2 + y^2 = (x - y)^2 \pmod{2}$, from which all statements in the corollary follow. Here again, $t$ may be a set of indeterminates, rather than a single one.

* The term $\alpha t_1^{e_1} t_2^{e_2} \cdots$ precedes $\beta t_1^{f_1} t_2^{f_2} \cdots$ if the degree of the first term exceeds the degree of the second term, or if the degrees are the same and the first non-zero number in the sequence $e_1 - f_1, e_2 - f_2, \cdots$ is positive.

Since there exist non-constant $2\times 2$ orthogonal matrices over $C_2[t]$, there exist also non-constant $r\times r$ orthogonal matrices over $C_2[t]$. Some of these can be obtained by bordering $2\times 2$ orthogonal matrices; others are products of such bordered matrices. I do not know whether there are still other $r\times r$ orthogonal matrices over $C_2[t]$.

**4. Further questions.** Questions concerning homogeneous polynomials in more than two variables, analogous to the questions considered in this note when the number of variables is 2, are not amenable to the same method of attack. The congruences below seem to show that the situation there is more complicated:

$$(t^3+t^2+t)^2+(t^2-1)^2=(t^3+t^2+1)^2 \pmod 3,$$

$$(2t^2+2t+1)^2+(t^2+t+1)^2+(2t+1)^2=3 \pmod 5.$$

## SOME PROPERTIES OF D NUMBERS

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**1. Introduction.** The well-known Fermat Theorem states that for any positive integer $a$ which is not divisible by the odd prime $p$, we have $a^{p-1}\equiv 1 \pmod p$. In 1912, R. D. Carmichael* proved the existence of composite positive integers $N$ which have the property that for any positive integer $a$ which is relatively prime to $N$, we have $a^{N-1}\equiv 1 \pmod N$. This is known as the *Fermat property* and such integers have been called *F numbers*.

An integer $N$, such as 195, 399 or 1023, which is greater than 3 and which has the property that for any positive integer $a$ which is relatively prime to $N$, we have $a^{N-3}\equiv 1 \pmod N$, we shall call a *D number*.

**2. Theorem.** We shall first establish the following result.

THEOREM 1. *If $N$ is a D number, it must be of one of the following forms:*

$$9p_1p_2\cdots p_s, \qquad 3p_1p_2\cdots p_s, \qquad p_1p_2\cdots p_t,$$

*where the $p_i$ are distinct primes $>3$, $s\geqq 1$, and $t>2$.*

*Proof.* First we note that since every prime possesses a primitive root, $N$ cannot be a prime. If $N=2^t$, where $t\geqq 2$, then

$$3^{N-3}-1=3^{2^t-3}-1=(3-1)(3^{2^t-4}+3^{2^t-5}+\cdots+1).$$

The second factor of this product is the sum of an odd number of odd terms. Hence $3^{N-3}-1$ is not divisible by 4. It follows that $N$ must contain an odd prime factor. Suppose $N=rp^k$, where $p$ is an odd prime and $r$ is not divisible by $p$. We may choose $s$ so that $x=w+sp^k$, where $w$ is a primitive root of $p^k$, is a prime $>N$. Then $x$ is relatively prime to $N$ and $x^{N-3}\equiv w^{N-3}\equiv 1 \pmod{p^k}$. Since $w$ is a primitive root of $p^k$, $N-3$ is divisible by $\phi(p^k)=p^k-p^{k-1}$. But $N-3$ is relatively prime to $p$ unless $p=3$. It follows that if $p\neq 3$, then $k=1$ and $N-3$

* This MONTHLY, vol. 19 (1912), pp. 22–27.

is divisible by $p-1$; and, since $p-1$ is even, $N$ must be odd. If $p=3$, then $N-3$ is divisible by $2\cdot 3^{k-1}$, so that $k\leqq 2$ and $r$ is odd.

Next suppose $N=p_1p_2$, where $p_1>p_2$ are primes $>3$. Then $N-3$ is divisible by $p_1-1$. Also $N-3\equiv p_1p_2-3\equiv p_2-3 \pmod{p_1-1}$. Hence $p_2-3$ is divisible by $p_1-1$, contrary to the supposition that $p_1>p_2$. Hence $N$ cannot be of the form $p_1p_2$.

**3. Additional results.** We have the following additional results concerning $D$ numbers.

THEOREM 2. *An integer $N$ which satisfies the conditions of Theorem* 1 *is a $D$ number if and only if $N-3$ is divisible by $\lambda(N)$, where $\lambda(N)$ is the minimal universal exponent* (mod $N$).

*Proof.* Since the primes $p_i$ have primitive roots, $N-3$ is a multiple of $p_i-1$. If $N$ is a multiple of 3, then $N-3$ is divisible by 3. Hence $N-3$ is a multiple of $\lambda(N)$. Conversely, if $N-3$ is divisible by $\lambda(N)$ then for $a$ relatively prime to $N$, we have

$$a^{N-3}=\left[a^{\lambda(N)}\right]^{N-3/\lambda(N)}\equiv 1 \pmod{N}.$$

THEOREM 3. *If $N=3^t\cdot p$, where $p$ is a prime $>3$, then $N$ is a $D$ number for every $p$ if $t=1$; but for $t=2$ it is a $D$ number only if $p=7$.*

*Proof.* If $N=3p$ where $p$ is a prime $>3$, then $N-3=3(p-1)$ and $\lambda(N)=p-1$. Hence $N-3$ is divisible by $\lambda(N)$. If $N=9p$ where $p$ is a prime $>3$, then $N$ is a $D$ number only if $9p-3$ is a multiple of $\lambda(9p)=[6,\ p-1]$. This is true only if $p=7$.

THEOREM 4. *If $N=p_1p_2p_3$, where $p_1$, $p_2$, and $p_3$ are distinct odd primes each $>3$, is a $D$ number, then the G.C.D. of $p_i-1$ and $p_j-1$ is a divisor of $p_k-3$.*

*Proof.* Let $d$ be the G.C.D. of $p_i-1$ and $p_j-1$. Then $p_ip_j\equiv 1 \pmod{d}$ and $N=p_ip_jp_k\equiv p_k\equiv 3 \pmod{d}$ since $N-3$ is divisible by $d$.

THEOREM 5. *If $N=p_1p_2p_3$, where $p_1$, $p_2$ and $p_3$ are distinct odd primes each $>3$, is a $D$ number, then either all $p_i$ are of the form $12k+11$ or one of them is of this form and two are of the form $12k+5$.*

*Proof.* If $p_1\equiv 1 \pmod 4$, then $N-3$ is divisible by $\lambda(N)$ only if $p_2p_3\equiv 3 \pmod 4$, and this is true only if $p_2\not\equiv p_3 \pmod 4$. In other words, either all $p_i$ are of the form $4k+3$ or exactly two of them are of the form $4k+1$. Also if $p_1\equiv 1 \pmod 6$, then $N-3\equiv 0 \pmod{\lambda(N)}$ only if $p_2p_3\equiv 3 \pmod 6$. This is impossible. Hence no $p_i$ is of the form $6k+1$.

THEOREM 6. *If $N=9p_1p_2$, where $p_1$ and $p_2$ are distinct odd primes each $>3$, is a $D$ number, then the G.C.D. of $p_1-1$ and $p_2-1$ is a divisor of* 6.

*Proof.* Let $d$ be the G.C.D. of $p_1-1$ and $p_2-1$. Then $p_1p_2\equiv 1 \pmod{d}$ and $N-3\equiv 6 \pmod{d}$. But $N-3$ is divisible by $d$. Hence $d$ is a divisor of 6.

## CLASSROOM NOTES

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### ANOTHER VARIATION OF NEWTON'S METHOD

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**1. Introduction.** Hamilton [1] has recently presented a generalization of Newton's Method of which the formula of Frame [2] and Wall [3] is a special case. The recent index of the MONTHLY [4] reveals a considerable literature on this topic, as well as some duplication. For example, Frame and Wall rediscovered in 1944–48 the method known to Halley [5] in 1694 and published in substantially the same form by Kobald in 1891. The general theory of successive approximation formulae has been discussed by Ford [7] and by Hamilton [8].

It is thus with some misgivings that one ventures to add to a literature well within the grasp of students who have completed an elementary course in the calculus. However, a variation of Newton's method which involves a very simple method of estimating the error in any approximation might justify itself.

**2. A Review.** The equation $f(x)=0$ has a root $x=r$ if $f(r)=0$. For $f(x)$ regular at $x=x_1$, we have, by Taylor's expansion:

$$(1) \qquad f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2!}(x - x_1)^2 + \cdots .$$

If we write $a_n$ for $f^{(n)}(x_1)/n!$, then

$$(1') \quad f(x) = a_0 + a_1(x - x_1) + a_2(x - x_1)^2 + \cdots + a_n(x - x_1)^n + \cdots .$$

The curve whose equation is

$$(2) \qquad y = a_0 + a_1(x - x_1) + a_2(x - x_1)^2 + \cdots + a_{k-1}(x - x_1)^{k-1}$$

has contact with the graph of $y=f(x)$ of order $k$, and the problem of determining a $k$-order iteration formula is that of finding the intercept $x_2$ of (2) to within powers of $(x_2-x_1)^k$. But this is identical with the process called inversion (or reversion) of series [9]. By putting $x=r$ and $y=f(r)=0$ in (2) then, when a finite number of terms in the solution of (2) are retained, $r$ becomes the second approximation $x_2$. Thus

$$(2') \qquad a_0 + a_1(x_2 - x_1) + a_2(x_2 - x_1)^2 + \cdots + a_{k-1}(x_2 - x_1)^{k-1} = 0.$$

Newton's method follows at once from (2') with $k=2$:

$$a_0 + a_1(x_2 - x_1) = 0,$$

$$x_2 = x_1 - a_0/a_1.$$

Frame's method follows from $k=3$, and (2) is geometrically an osculating parabola. The approximate intercept was found by writing

$$(3) \qquad (x_2 - x_1) = \frac{-a_0}{a_1 + a_2(x_2 - x_1)}$$

and using Newton's value $-a_0/a_1$ for $(x_2-x_1)$ on the right:

$$x_2 = x_1 - \frac{a_0 a_1}{a_1^2 - a_0 a_2}.$$

Hamilton solves (3) by constructing from (2′) the system:

$$(4) \qquad \begin{aligned} a_1(x_2 - x_1) + a_2(x_2 - x_1)^2 &= -a_0 - a_3(x_2 - x_1)^3 - \cdots \\ a_0(x_2 - x_1) + a_1(x_2 - x_1)^2 &= -a_2(x_2 - x_1)^3 - \cdots . \end{aligned}$$

Since $k=3$, we neglect powers of $(x_2-x_1)^k$, $k \geqq 3$, and solve by Cramer's rule:

$$x_2 - x_1 = \frac{\begin{vmatrix} -a_0 & a_2 \\ 0 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} = \frac{-a_0 a_1}{a_1^2 - a_0 a_2}.$$

Hamilton generalizes the system (4) to obtain iterative formulas of any desired order.

**3. Another Variation.** The approximate error in a $k$-order formula is given by the term in $(x_2-x_1)^k$ in the solution of (2). In Hamilton's system corresponding to (4), this is found by retaining one more term on the right of each of the $k$ equations. This means that each of the equations contributes to the error. We can avoid this by constructing the system from (2′) in such a way that the error enters only once:

$$(5) \qquad \begin{aligned} &a_1(x_2 - x_1) + a_2(x_2 - x_1)^2 \\ &\quad + \cdots + a_{k-1}(x_2 - x_1)^{k-1} && = -a_0 - a_k(x_2 - x_1)^k - \cdots \\ &a_0(x_2 - x_1) + a_1(x_2 - x_1)^2 \\ &\quad + \cdots + a_{k-2}(x_2 - x_1)^{k-1} + a_{k-1}(x_2 - x_1)^k && = -a_k(x_2 - x_1)^{k+1} - \cdots \\ &a_0(x_2 - x_1)^2 \\ &\quad + \cdots + a_{k-3}(x_2 - x_1)^{k-1} + a_{k-2}(x_2 - x_1)^k && = -a_{k-1}(x_2 - x_1)^{k+1} - \cdots \\ &\quad \cdots && = \qquad \cdots \end{aligned}$$

Let the principal determinant be

$$(6) \qquad D_k = \begin{vmatrix} a_1 & a_2 \cdots a_{k-1} & 0 \\ a_0 & a_1 \cdots a_{k-2} & a_{k-1} \\ 0 & a_0 \cdots a_{k-3} & a_{k-2} \\ 0 & 0 \cdots a_0 & a_1 \end{vmatrix}$$

and let $M_k$ be the minor of the upper left-hand element of $D_k$, then

$$x_2 - x_1 = \frac{-[a_0 + a_k(x_2 - x_1)^k]M_k}{D_k}.$$

If $x_2$ is replaced by $x+\epsilon_2$ and $x_1$ by $x+\epsilon_1$, then the lowest powers of $\epsilon_1$ in the expression $a_kM_k/D_k$ come from the principal diagonals, and we have

$$(7) \qquad x_2 = x_1 - \frac{a_0M_k}{D_k} - \frac{a_k}{a_1}(x_2 - x_1)^k.$$

When $a_k=0$, $k\geqq 3$, the error term reduces to $(-a_2/a_1)^k(x_2-x_1)^{k+1}$.

*A numerical example for illustration.* A suitable first approximation may be found by any of many methods of estimating. To calculate $\sqrt{2}$, we may start with $x_1=7/5$, then $a_0=-1/25$, $a_1=14/5$, $a_2=1$, $a_k=0$, $k\geqq 3$. Then with $k=5$,

$$x_2 = \frac{7}{5} + \frac{7801}{548842},$$

and before reducing to decimal form we may determine the number of places to retain in the division. The error is

$$E \sim (-5/14)^5(.014)^6 = -4.37(10^{-14}).$$

Then

$$\begin{aligned} x_2 &= 1.4142135623731419 \\ &\quad - \qquad\qquad 437 \\ &= 1.4142135623730982 \end{aligned}$$

which we know to be true for fourteen places.

**4. Conclusion.** The method works equally well for transcendental equations, and for imaginary as well as real roots. As in the derivation of other iterative formulas, there are several approaches. For example, the case $k=3$ follows from two applications of Newton's method with terms involving derivatives of higher order than the second being omitted; or again, by using Frame's formula instead of Newton's in (3).

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## A SIMPLE PROOF OF STIRLING'S FORMULA

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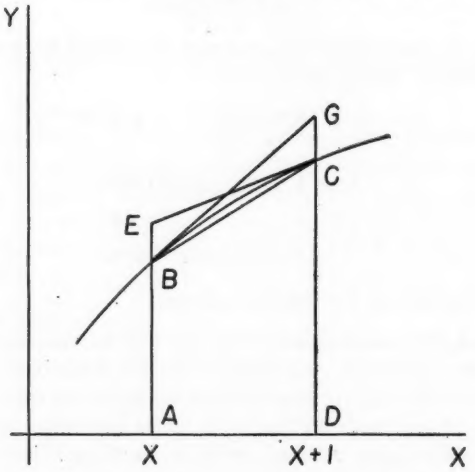
Stirling's formula for the asymptotic value of factorial $N$ is widely used in physics and many different proofs of it are known. The following, which is a slight variation on an old approach, is so simple as to be useful as an exercise in integration and the convergence of montone, bounded sequences. It could therefore be used in a first course in Calculus.

Consider the graph of $y=\ln x$, where by $\ln x$ we mean the natural logarithm of $x$. Since $y'=1/x$, the slope between $x$ equal 1 and $N$ is positive and decreas-




ing. Thus chords, such as $BC$ in the accompanying figure, are below the curve and therefore the area of the trapezoid $ABCD$ is less than the area under the curve. The area of the trapezoid is

$$\tfrac{1}{2}[\ln x + \ln (x+1)],$$

and therefore the area of the $N-1$ trapezoids from $x$ equal 1 to $N-1$ is

$\frac{1}{2}\{[\ln 1+\ln 2]+[\ln 2+\ln 3]+\cdots+[\ln(N-1)+\ln N]\}=\ln N!-\frac{1}{2}\ln N.$

Thus

$$(1)\qquad c_N=\int_1^N \ln x\,dx-\ln N!+\tfrac{1}{2}\ln N=(N+\tfrac{1}{2})\ln N-N+1-\ln N!$$

is a positive number equal to the area under the curve $y=\ln x$ from $x=1$ to $x=N$ minus the trapezoidal approximation to this area. The sequence $c_N$ therefore increases monotonely with $N$.

On the other hand, the area under the curve is clearly less than the average of the areas of the trapezoids $AECD$ and $ABGD$, where $BG$ is the tangent to the curve at $B$ and $CE$ is the tangent at $C$. From the known value of the slope, we have $DG$ equal to $\ln x+1/x$ and $AE$ equal to $\ln(x+1)-1/(x+1)$. Thus the average area of the two trapezoids is

$$\frac{1}{2}\left[\ln x+\ln(x+1)+\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+1}\right)\right].$$

Summing as before, we have

$$\int_1^N \ln x\,dx<\ln N!-\frac{1}{2}\ln N+\frac{1}{4}-\frac{1}{4N},$$

and, therefore,

$$c_N<\frac{1}{4}-\frac{1}{4N}<\frac{1}{4}.$$

Thus $c_N$, which is a monotone increasing sequence, is bounded above and therefore has a limit $c$ such that $0\leqq c\leqq\frac{1}{4}$. If we set $e^{1-c}=a$, so that $2.11<a<2.72$, it follows from (1) that

$$(2)\qquad \frac{N!}{\sqrt{N}\,N^N e^{-N}}\to a$$

as $N$ tends to infinity. This is the essential part of Stirling's formula. However, we must still find the value of $a$. By geometrical intuition our method of approximation from below seems about twice as accurate as the approximation from above, so we shall expect $a$ to be closer to 2.72 than to 2.11, perhaps around 2.5.

There is a well-known trick for finding the exact value of $a$, by making use of Wallis' formula. To prove the latter, we set $I_r=\int_0^{\pi/2}\sin^r\phi\,d\phi$, and since in the range $0<\phi<\pi/2$,

$$\sin^{2k-1}\phi>\sin^{2k}\phi>\sin^{2k+1}\phi$$

we have

$$(3)\qquad I_{2k-1}>I_{2k}>I_{2k+1}.$$

Integrating $I_r$ by parts, one easily shows that $rI_r=(r-1)\,I_{r-2}$, and therefore $I_r$ may be evaluated by recursion. The inequalities (3) then immediately give Wallis' Formula:

$$\frac{[2^k(k!)]^4}{[(2k)!]^2(2k+1)}\to\frac{\pi}{2}$$

as $k$ approaches infinity.* But by (2) the left-hand side of this approaches $a^2/4$. Thus $a=\sqrt{2\pi}\doteq 2.50$, and we finally have:

$N!$ is asymptotically equal to $\sqrt{2\pi N}\,e^{-N}N^N$.

## BRINGING IN DIFFERENTIALS EARLIER

W. R. RANSOM, Tufts College

Suppose we skip the usual chapter on functions and continuity, and start with the concept of variables connected by equations:

$$x^2+y^2=r^2,\qquad x=r\cos\theta,\cdots. \tag{1}$$

Not variables as "symbols to which numerical values can be assigned," but variables as Newton (and beginners in calculus) might conceive them—the measures of quantities in nature and geometry. And not continuity as unnaturally defined for a point only, but the sort of continuity implied in the ancient doctrine "natura non agit per saltum."

A set of variables connected by equations will have sets of simultaneous numerical values, and such a set constitutes a "state." In an interval from one state to another, the variables receive increments (denoted by $\Delta x, \Delta y, \cdots$ *etc.*) and the new values, $x+\Delta x, \cdots$ are connected by the same relations as before. The intervals between states are taken small enough so that no question of multiple values arises.

By subtractions a new set of equations is obtained which describe relations between the variables and their increments. If these equations are all divided by the same increment, we get relations between the variables for some state and some mean rates of increase for an interval from that state to another. We define exact rate (now simply called rate) as the limit which the mean rate approaches when the interval shrinks so that the second state approaches the first. These rates are then adjoined to the set of variables, thus enlarging the state.

Now we come to differentials. In the equations which connect the variables with the rates, express the rates as fractions with a common denominator. This denominator is called the differential of that variable whose increment was used as a divisor, and the numerators are called the differentials of the variables in whose rates they appear. That is, we have symbolized the rates so that $D_xy=dy/dx$, $D_xr=dr/dx$, $\cdots$. This defines the differentials, one as a denominator

* See Courant and Robbins, *What is Mathematics?*, pp. 509–510, for details.

of rates, and the others as numerators of rates. Now adjoin these differentials to the state containing variables and rates.

By the argument which appears in current texts under the title "function of a function," it is then to be shown that not only are

$$D_x y = dy/dx, \qquad D_x r = dr/dx, \cdots$$

but also:

$$D_y x = dx/dy, \qquad D_r y = dy/dr, \cdots .$$

Accordingly the distinction relating to one differential as denominator of rates may, and should forthwith be laid aside: any two differentials in a state may be taken as numerator and denominator for the corresponding rate.

It is desirable to frame rules for writing down the equations that connect variables with their differentials, as, from (1):

$$2xdx + 2ydy = 2rdr, \; dx = dr \cos\theta + r \sin\theta d\theta, \cdots .$$

From these we get the information about $dy/dx$, $dr/dy$, $d\theta/dy$, $\cdots$.

The definition commonly given in current texts for $dy$ is $dy = (D_x y)\Delta x$, whence $dx = \Delta x$. This does not make sense, since $\Delta x$ is not defined for the state to which the *rates* belong, but only for the interval to which mean rates belong, Graphically $dy$ and $dx$ are the rise and run of a piece of the tangent, and have nothing to do with two points on the curve.

Since differentials are determined numerically only to within a common factor, they cannot themselves be differentiated. There is no $d^2y$. The symbol $d^2y/dx^2$ is shorthand for $d(dy/dx)/dx$ just as $D_x^2 y$ is shorthand for $D_x(D_x y)$.

By using differentials from the start, we get rid of the clumsy phrase "with respect to" in both differentiation and integration. The $dx$ in the integral tables becomes a differential, a factor of the differential to be integrated, and not a part of a symbol "$\int(\ )dx$." It seems scandalous to the writer that the $dx$ in $dy/dx$ should be equal to $\Delta x$, while the $dx$ in $\int dx$ should not be equal to $dx$.

*Editorial Note*: This paper by Professor Ransom is the outgrowth of extended discussion between him and the editor arising from a discussion group at Tufts College in August, 1950 on the "Teaching of Calculus." Although the editor does not agree with Professor Ransom at many points, he is publishing this paper as a stimulus to discussion of an area in which there seems to be great confusion among textbook writers and teachers. Classroom Notes will welcome a limited number of replies to Professor Ransom's paper, particularly those which discuss the questions:

(1) What is a differential?

(2) Does the "$dx$" of differential calculus have the same meaning as the "$dx$" in the integral: $\int_a^b f(x)\,dx$?

Prospective commentators will be interested in the various articles on this subject in earlier volumes of this MONTHLY. These are listed under "Differentiation" in the Index which was recently issued to readers of the MONTHLY.

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 966. *Proposed by C. W. Bruce, Tennessee Polytechnic Institute.*

Any integer $N$ may be written in the form $2^k(a^2-b^2)$, where $k$, $a$, $b$ are non-negative integers. When are $k$, $a$, $b$ unique? When not unique, how many sets of $k$, $a$, $b$ are there?

E 967. *Proposed by L. A. Ringenberg, Eastern Illinois State College.*

Let $P_1, P_2, \cdots, P_n$ $(n>4)$ be $n$ points equally spaced and arranged in order of subscripts on a circle of radius 1, and let $S_n$ be the area of the star formed by the line segments $P_1P_3, P_2P_3, \cdots, P_{n-1}P_1, P_nP_2$. (a) Find a formula for $S_n$. (b) Find the smallest positive integer $n$ such that $nS_n>(n-1)\pi$. (c) Show that if $k$ is any positive integer greater than 1, then $n^kS_n<(n^k-1)\pi$ for $n=5, 6, 7, \cdots$.

E 968. *Proposed by R. M. Redheffer, University of California.*

If $f(s)=\int_{-\pi}^{\pi} e^{\cos x}\cos(\sin x)\cos sx\,dx$, find $\sum_1^\infty f(n)$.

E 969. *Proposed by Solomon Golomb, The Johns Hopkins University.*

Show that a necessary and sufficient condition that there be infinitely many twin primes is that there be infinitely many numbers $n$ not of the form $6ab\pm a\pm b$.

E 970. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Show that in triangle $ABC$ side $BC$ bisects the segments intercepted by the other two sides on the tangents to the inscribed parabola having for focus the point $A_1$ where the symmedian $AA_1$ intersects the circumcircle.

### SOLUTIONS

#### Integers Less Than $N$ as Sum of Divisors of $N$

E 932 [1950, 556]. *Proposed by Paul Erdös, University of Illinois.*

(1) Prove that every positive integer not exceeding $n!$ is the sum of $n$ or fewer distinct divisors of $n!$.

(2) Denote by $E(n)=2^{a_2}3^{a_3}\cdots p^{a_p}$ the least common multiple of the positive integers not exceeding $n$. Prove that every positive integer not exceeding $E(n)$ is the sum of $a_2+a_3+\cdots+a_p$ or fewer distinct divisors of $E(n)$.

*Solution by Joseph Oppenheim, Des Plaines, Illinois.* (1) Let us assume the theorem is true for $n=k$, and let $x$ be an integer not exceeding $(k+1)!$. We can

write $x=p(k+1)+q$, where $q<k+1$. Then

$$(k+1)k! \geqq x \geqq p(k+1),$$

or $p \leqq k!$. By the induction hypothesis $p=a_1+a_2+\cdots+a_m$, where $a_i \neq 0$, $a_i \mid k!$, $m \leqq k$, and $a_i \neq a_j$ if $i \neq j$. Hence

$$x = (k+1)\sum_1^m a_i + q = \sum_1^m (k+1)a_i + q.$$

The number of summands is $m+1 \leqq k+1$. Since $q<k+1$, $q \neq (k+1)a_i$ for all $i$. Further, since the $a_i$'s are distinct, so are all the $a_i(k+1)$'s.

Finally, for $k=1, 2, 3$ the theorem is obvious.

(2) Let us make the notation a little more explicit by setting

$$E(n) = LCM\{1, 2, \cdots, n\} = 2^{a_2(n)}3^{a_3(n)} \cdots p^{a_p(n)}$$

and

$$E^*(n) = a_2(n) + a_3(n) + \cdots + a_p(n).$$

We first establish the

LEMMA: $J(n) \equiv E^*(n) - E^*(n-1) = 0$ *or* 1. *If* $J(n)=1$ *then* $n = p^{a_p(n)}$ and $E(n) = pE(n-1)$. *If* $J(n)=0$ *then* $E(n) = E(n-1)$.

Clearly $E(n)$, $E^*(n)$, and $a_p(n)$ for $p=2, 3, 5, \cdots$, are all monotonically increasing functions. If $J(n) \neq 0$ there exists a $p$ such that $a_p(n-1) < a_p(n)$. Now

$$p^{a_p(n-1)} \leqq n-1 < p^{a_p(n-1)+1} \leqq p^{a_p(n)} \leqq n.$$

It follows that

$$p^{a_p(n-1)+1} = p^{a_p(n)} = n.$$

Therefore $E(n) = pE(n-1)$ and $p$ is the only prime whose exponent is altered. Hence $J(n)=1$. If $J(n)=0$, then $a_p(n) - a_p(n-1) = 0$ for all $p$, and $E(n) = E(n-1)$.

We now prove the theorem by induction. We assume, for any $k \leqq n$, that $x \leqq E(k)$ implies that $x$ is the sum of at most $E^*(k)$ distinct divisors of $E(k)$.

Take $x \leqq E(n+1)$ and let $m$ be the least integer such that $E(m) = E(m+1)$. By the lemma we have $m = p^{ap(m)}$ and $E(m) = pE(m-1) = E(n+1)$. Now $x = qp + r$, where $r < p$ and $q \leqq E(m-1)$ and $m-1 \leqq n$. Hence $q \sum_1^h \alpha_i$, where $\alpha_i \neq \alpha_j$ for $i \neq j$, each $\alpha_i \mid E(m-1)$, and $h \leqq E^*(m-1)$. Therefore $x = \sum_1^h p\alpha_i + r$. Now $r < p \leqq m$, whence $r \mid E(m) = E(n+1)$. Also, $r \neq p\alpha_i$ and $p\alpha_i \mid pE(m-1) = E(m) = E(n+1)$. Finally, $h+1 \leqq E^*(m-1)+1 = E^*(m) = E^*(n)$. Since the theorem is obviously true for $n=1, 2, 3$, it now follows for all $n$.

Also solved by Leopold Flatto, Roger Lessard, and Fred Marer.

**An Interesting Function**

E 935 [1950, 557]. *Proposed by C. D. Olds, San Jose State College*

Given an example of a function which is discontinuous in an everywhere dense set, and which is also differentiable in an everywhere dense set.

*Solution by J. W. Gaddum, University of Missouri.* We define $f(x)$ for $x \epsilon [0, 1]$. If $x = k/2^n$, where $k$ is an odd integer less than $2^n$, set $f(x) = 1/2^{2n-1}$. Otherwise set $f(x) = 0$.

For $2^{n-1}$ values of $x$, $f(x) = 1/2^{2n-1}$. Hence the variation of $f(x)$ on $[0, 1]$ is

$$\sum_{n=1}^{\infty} 2^{n-1}/2^{2n-1} = \sum_{n=1}^{\infty} 1/2^n = 1/2.$$

Hence $f(x)$ is of bounded variation and is differentiable almost everywhere. But $f(x)$ is discontinuous for $x = k/2^n$.

Also solved by Leopold Flatto, Norman Miller, Leo Moser, C. S. Ogilvy, L. A. Ringenberg, J. B. Rosser, O. E. Stanaitis, Albert Wilansky, and the proposer.

Several solutions offered $f(x) = 0$ if $x$ is irrational and $f(x) = q^{-q}$ if $x = p/q$, $(p, q) = 1$. Then $f(x)$ is discontinuous for every rational $x$ and differentiable for every algebraic irrationality. A very elementary, but perhaps somewhat unsportsmanlike, example is $f(x)$ defined on $[-1, 1]$ by $f(x) = 1$ for negative rational $x$ and $f(x) = 0$ otherwise.

F. Bagemihl located this problem, with a solution, in K. Knopp, *Aufgabensammlung zur Funktionentheorie*, vol. 2, Berlin (1949), p. 9, prob. 2; p. 55.

**The Scratched Lawn Edger**

E 936 [1950, 632]. *Proposed by R. E. Horton, Los Angeles City College.*

A wheel of radius $a$ with a flange of radius $b$ is rolling without slipping along a straight rail. On the corner of the rail a piece projects which scratches a mark on the face of the flange. Identify the curve of the mark so formed. What conditions on $a$ and $b$ are necessary in order that the mark on the flange will be a simple closed curve?

*Solution by C. S. Ogilvy, Columbia University.* Considering the wheel fixed and the rail rotating around it, the mark is at once identified as two arcs of involutes of the wheel.

For a simple closed curve, the point $(-b, 0)$ satisfies the parametric equations of the involute, giving

$$-b = a(\cos\phi + \phi \sin\phi), \qquad 0 = a(\sin\phi - \phi\cos\phi).$$

Squaring and adding we find that $b/a = (1+\phi^2)^{1/2}$, where, from the second equation, $\phi$ satisfies the relation $\phi = \tan\phi$. An approximate numerical value for $b/a$ is 4.60.

Also solved by Alan Berndt, E. A. Franz (partially), and the proposer.

**A Generalization of Wilson's Theorem**

E 937 [1950, 632]. *Proposed by I. N. Herstein, University of Kansas.*

If $p$ is a prime and $n \geqq p$, then

$$n! \sum_{pi+j=n} 1/p^i i!j! \equiv 0, \bmod p.$$

*Solution by Kirk Stewart, College of Puget Sound.* We observe that

$$n! \sum_{pi+j=n} 1/p^i i!j! = \sum_{i=0}^{[n/p]} n!/p^i i!(n - ip)!.$$

Now consider

$$n!/p^i i!(n - ip)! = (n - ip + 1)(n - ip + 2) \cdots (n - 1)n/p^i i!.$$

In the numerator on the right side there are: (a) $ip$ factors, and hence $i$ complete systems of residues modulo $p$, (b) $i$ multiples of $p$, namely $p[n/p]$, $p([n/p]-1)$, $\cdots$, $p([n/p]-i+1)$. Denote a complete system of residues by $\rho_j$, $j=0, 1, \cdots, p-1$, with $\rho_0 \equiv 0$, mod $p$. Then

$$\prod_{j=1}^{p-1} \rho_j \equiv (p - 1)! \equiv -1, \bmod p,$$

by Wilson's theorem, and consequently

$$\begin{aligned} n!/p^i i!j! &\equiv [n/p]([n/p] - 1) \cdots ([n/p] - i + 1)(-1)^i/i! \\ &\equiv \binom{[n/p]}{i} (-1)^i, \bmod p. \end{aligned}$$

It follows that

$$n! \sum_{pi+j=n} 1/p^i i!j! \equiv \sum_{i=0}^{[n/p]} (-1)^i \binom{[n/p]}{i} = (1 - 1)^{[n/p]} \equiv 0, \bmod p.$$

Also solved by B. W. Brewer, Roger Lessard, Azriel Rosenfeld, and the proposer.

**Euler's Constant**

E 938 [1950, 632]. *Proposed by H. F. Sandham, Trinity College, Ireland.*

If $x = \sum_{i=1}^{n} 1/i$, prove that $n = [e^{x-\gamma}]$, where $\gamma$ is Euler's constant, and $[a]$ denotes the integral part of $a$.

*Solution by S. H. Gould, Purdue University.* If

$$f(y) = 1/(y + 1) - \log (y + 1)/y,$$

then $f(1) < 0$, $f'(y) > 0$, $\lim_{y \to \infty} f(y) = 0$, so that $f(y) < 0$ for all $y \geqq 1$. Thus $x - \log n$

is monotone decreasing. Similarly, $x-\log(n+1)$ is monotone increasing. Thus $\log n<x-\gamma<\log(n+1)$, or $n<e^{x-\gamma}<n+1$.

Also solved by Philip Anselone and Vern Hoggatt (jointly), P. T. Bateman, D. H. Browne, Leopold Flatto, P. G. Kirmser, M. S. Klamkin, Norman Miller, Leo Moser, C. F. Pinzka, Alex Rosenberg, O. E. Stanaitis, and the proposer.

### A Chain of Circles

E 939 [1950, 632]. *Proposed by P. J. Schillo, University of Buffalo.*

Circles $C_0, C_1, C_2, \cdots$, having radii $r_0, r_1, r_2, \cdots$, are tangent to a circle $C$, of radius $2r$, and to a diameter of $C$. Circle $C_0$ passes through the center of $C$, and $C_n$ is tangent externally to $C_{n-1}$. Show that

$$r_n = 4w^n r/(w^n+1)^2,$$

where $w=3+2\sqrt{2}$, and that $r/r_n$ is an integer.

*Solution by W. O. Pennell, Exeter, N. H.* Take the diameter of $C$ as $x$-axis and the center of $C$ as origin. Then it is easily found that the coordinates of the centers of $C_k$ and $C_{k+1}$ are

$$(r_k,\ 2(r^2-rr_k)^{1/2}) \quad \text{and} \quad (r_{k+1},\ 2(r^2-rr_{k+1})^{1/2}).$$

Since the distance between these centers must be $r_k+r_{k+1}$ we have

$$f(r_k, r_{k+1}) \equiv (r_k+r_{k+1})^2-(r_k-r_{k+1})^2-4\left[(r^2-rr_k)^{1/2}-(r^2-rr_{k+1})^{1/2}\right]^2=0.$$

Now, for a given value of $r_k$, there is a unique value of $r_{k+1}$ such that $f(r_k, r_{k+1})=0$ and $r_{k+1}<r_k$. By substitution we find that the values

$$r_k = 4w^k r/(w^k+1)^2, \qquad r_{k+1} = 4w^{k+1}r/(w^{k+1}+1)^2,$$

where $w=3+2\sqrt{2}$, are such that $f(r_k, r_{k+1})=0$ and $r_{k+1}<r_k$. Therefore if $r_k$ has the value given above, then $r_{k+1}$ must also have the value given above. But we do have $r_0=2r=4w^0r/(w^0+1)^2$. Hence the first part of the problem follows by mathematical induction.

We next observe that

$$\begin{aligned} r/r_n = (w^n+1)^2/4w^n &= \left[(3+2\sqrt{2})^n+2+(3-2\sqrt{2})^n\right]/4 \\ &= \left[2(3^n+1)+\cdots\right]/4, \end{aligned}$$

where the unwritten terms contain even powers of $2\sqrt{2}$ as factors. It follows that $r/r_n$ is an integer.

Also solved by A. L. Epstein, Vern Hoggatt, M. S. Klamkin, Roger Lessard, O. Dale Smith, and the proposer. Smith solved the first part of the problem by the method of inversion, using one end of the diameter of $C$ as center of inversion.

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4438. *Proposed by H. F. Sandham, Trinity College, Ireland.*

Show that the polynomial of degree $2n$ defined by the hypergeometric function $F(-2n, \frac{1}{2}; n+1; x)$ takes on the value 1 when $x=4$.

4439. *Proposed by Josef Langr, Prague, Czecho-Slovakia.*

A line $m$ cuts the sides $BC$, $CA$, $AB$ of a triangle $ABC$ at $L$, $M$, $N$. Show that the midpoints of the segments $AL$, $BM$, $CN$ are on a line $n$, and find the locus of the point of intersection of $m$ and $n$ as $m$ moves parallel to itself.

4440. *Proposed by R. G. Stoneham, University of California, Berkeley.*

Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_{2k} \Pi^{2k}}{(2k+1)!}$$

where $B_{2k}$ are the even Bernoulli numbers.

4441. *Proposed by R. M. Cohn, Rutgers University.*

If a polynomial equation $f(x)=0$, with integral coefficients, has no positive roots, there exists a polynomial $g(x)$ with integral coefficients such that all coefficients of $f(x) \cdot g(x)$ are positive.

4442. *Proposed by Michael Golomb, Purdue University*

If a polynomial (not a constant) with complex coefficients is of the same modulus at two points $z_1$, $z_2$ of the $z$-plane, show that it has at least one zero in the open half-plane

$$R\{\bar{z}(z_1 - z_2)(|z_1| - |z_2|)\} > 0$$

if $|z_1| \neq |z_2|$, and at least one zero in each of the closed half-planes

$$R\{\bar{z}(z_1 - z_2)\} \geqq 0, \qquad R\{\bar{z}(z_1 - z_2)\} \leqq 0$$

if $|z_1| = |z_2|$.

## SOLUTIONS

### A Factor Problem

4339 [1949, 187]. *Proposed by Paul Erdös, University of Aberdeen, Scotland.*

Prove that if $n=2^{\mu}(2k+1)$, $k>1$ and $\mu$ being positive integers, then $2^n-1$ has a composite factor congruent to 1 modulo $n$.

*Solution by N. G. W. H. Beeger, Amsterdam, Holland.* We make use of the well known theorems:*

(1) $2^m+1$, $m\neq 3$, is divisible by a prime of the form $2mx+1$.

(2) $2^m-1$ is divisible by a prime of the form $mx+1$.

(3) $2^{4m}+1$ is divisible by a prime of the form $16mx+1$.

For $n=2^{\mu}(2k+1)$, $k>1$, $\mu>0$, we have

$$2^n - 1 = (2^d - 1)(2^d + 1)(2^{2d} + 1) \cdots (2^{2^{\mu-1}d} + 1), \tag{A}$$

in which we have put $d$ for the odd number $2k+1$. From (1) the last factor of (A) is divisible by a prime $X=nx+1$. From (3) with $m=2^{\mu-4}d$, $\mu\geqq 4$, the factor $2^{2^{\mu-3}d}+1$ is divisible by a prime $W=nw+1$. Hence $XW$ is the required composite factor and the theorem is proved for $\mu\geqq 4$.

If $\mu=1$, the right member of (A) has only two factors, the second of which is divisible by a prime $X=nx+1$ from (1). The first factor is divisible by a prime $dy+1$ from (2); hence this (odd) prime has the form $Y=2dy+1=by+1$. So $XY$ is the required composite factor.

If $\mu=2$, the right-hand member of (A) has three factors, the last of which is divisible by a prime $X=nx+1$ from (1). The first and second factors are divisible by primes $Y=2dy+1$, $Z=2dz+1$, respectively. If $y$ or $z$ is even the theorem is proved with $XY$ or $XZ$; if $y$ and $z$ are both odd, then $YZ$ has the form $nu+1$, and the theorem is also proved.

If $\mu=3$, the right-hand side of (A) has four factors, the last of which is divisible by a prime $X=nx+1$ from (1). The other three factors are divisible respectively by primes $Y=2dy+1$, $Z=2dz+1$, $U=4du+1$. It is easily verified that the desired composite factors may be selected as follows:

| | | |
|---|---|---|
| $u \equiv 0 \pmod 2$, | $XU$; | |
| $y \equiv 0 \pmod 4$, | $XY$; | |
| $u \equiv 1 \pmod 2$, | $y \equiv z \pmod 4$, | $XYZ$; |
| $u \equiv 1 \pmod 2$, | $y \equiv 2 \pmod 4$, | $XZU$; |
| $u \equiv 1 \pmod 2$, | $y \equiv 1$, $z \equiv 3 \pmod 4$, | $XYZU$. |

Since $y$ and $z$ are interchangeable this gives all the cases and the proof is complete.

* These are corollaries of Bang's theorem. See Kraitchik, *Theorie des Nombres*, v. 1, p. 129; also Dickson, *History of the Theory of Numbers*, v. 1, chapt. xvi. (3) was noted by Lucas.

**A Sum of Reciprocals**

4365 [1949, 637]. *Proposed by Paul Erdös, University of Aberdeen, Scotland.*

Let $a_1<a_2<a_3< \cdots <a_k \leqq n$ be such that the least common multiple of any two $a$'s exceeds $n$. Prove that

$$\sum_{i=1}^{k} \frac{1}{a_i} < 2.$$

*Solution by R. S. Lehman, Stanford University.* The problem is trivial if $a_1=1$. Let us take $a_1 \geqq 2$ and let $b_m$ be the number of $a$'s satisfying the condition

$$\frac{n}{m} < a_i \leqq \frac{n}{m-1}. \tag{1}$$

Then exactly $b_m$ of the $a$'s have $(m-1)$ multiples each among the first $n$ integers, $1, 2, \cdots, n$; and no two $a$'s have a common multiple among these $n$ integers. Therefore

$$b_2 + 2b_3 + 3b_4 + \cdots + nb_{n+1} = n - r_1, \qquad r_1 \geqq 0. \tag{2}$$

For any given set of $a$'s, $a_1<a_2< \cdots <a_k$, we may, without loss of generality, choose $n$ to be equal to $a_k$. Then from (1) it follows that

$$\sum_{i=1}^{k} \frac{1}{a_i} < \frac{1}{n} + (b_2 - 1)\frac{2}{n} + b_3\frac{3}{n} + b_4\frac{4}{n} + \cdots + b_{n+1}\frac{n+1}{n}$$

$$= \frac{1}{n}(-1 + 2b_2 + 3b_3 + 4b_4 + \cdots + (n+1)b_{n+1}).$$

Substituting the value for $b_2$ from (2), we obtain

$$\sum_{i=1}^{k} \frac{1}{a_i} < 2 - \frac{1}{n} - \frac{2r_1}{n} - \frac{1}{n}(b_3 + 2b_4 + 3b_5 + \cdots + (n-1)b_{n+1}), \tag{3}$$

which proves the proposed result.

To improve this estimate, consider the integers $1, 2, 3, \cdots, [n/2]$. Of the $a$'s, $b_3+b_4$ have one multiple each among these integers; $b_5+b_6$ have two multiples each among them, and so forth. Then

$$b_3 + b_4 + 2b_5 + 2b_6 + 3b_7 + 3b_8 + \cdots = \left[\frac{n}{2}\right] - r_2, \qquad r_2 \geqq 0. \tag{4}$$

Further, if $p>[n/2]$ is not divisible by any of the $a$'s, we can include $p$ among the $a$'s, since any multiple of $p$ other than $p$ itself is greater than $n$. (Such an addition to the set of $a$'s merely increases $\sum 1/a_i$.) But $r_1-r_2$ is the number of integers which are greater than $[n/2]$ and less than $n$, and which are not multiples of $a$'s. If all such $p$'s have been included we have consequently $r_1-r_2=0$. Substituting the value of $b_3$ given by (4) into (3) and using the fact

that $r_1-r_2=0$ we have

$$\sum_{i=1}^{k}\frac{1}{a_i}<2-\frac{1}{n}-\frac{r_1}{n}-\frac{1}{n}\left[\frac{n}{2}\right]$$
$$-\frac{1}{n}(b_4+b_5+2b_6+2b_7+3b_8+\cdots)<1\tfrac{1}{2}. \tag{5}$$

Further consider the integers $1, 2, 3, \cdots, [n/3]$. Of the $a$'s, $b_4+b_5+b_6$ have one multiple each among these integers; $b_7+b_8+b_9$ have two, and so forth. Therefore

$$b_4+b_5+b_6+2b_7+2b_8+2b_9+3b_{10}+3b_{11}+\cdots=\left[\frac{n}{3}\right]-r_3,\ r_3\geqq 0. \tag{6}$$

Substituting from (6) into (5) we obtain

$$\sum_{i=1}^{k}\frac{1}{a_i}<2-\frac{1}{n}-\frac{r_1}{n}-\frac{1}{n}\left[\frac{n}{2}\right]-\frac{1}{n}\left[\frac{n}{3}\right]+\frac{r_3}{n}$$
$$-\frac{1}{n}(b_6+b_8+b_9+b_{10}+b_{11}+2b_{12}+\cdots),$$

where all the terms inside the parenthesis are positive. Since

$$\frac{1}{n}\left[\frac{n}{2}\right]\geqq\frac{1}{n}\left(\frac{n}{2}-\frac{1}{2}\right),\qquad\frac{1}{n}\left[\frac{n}{3}\right]\geqq\frac{1}{n}\left(\frac{n}{3}-\frac{2}{3}\right),$$

we have

$$\sum_{i=1}^{k}\frac{1}{a_1}<2-\frac{1}{n}-\frac{r_1}{n}+\frac{r_3}{n}-\frac{1}{2}-\frac{1}{3}+\frac{1}{2n}+\frac{2}{3n}.$$

Since $r_1$ is the number of integers $\leqq n$ that are not multiples of any $a$, and $r_3$ is the number of integers $\leqq[n/3]$ having the same property, it is clear that $r_1\geqq r_3$. It follows that

$$\sum_{i=1}^{k}\frac{1}{a_i}<\frac{7}{6}+\frac{1}{6n}.$$

As an easy corollary, $\sum 1/a_i<6/5$.

Also solved by G. F. D. Duff, A. W. Fuller, D. J. Newman, and the Proposer.

**Tetrahedron with Concentric Inscribed and Circumscribed Spheres**

4366 [1949, 637]. *Proposed by Joseph Rosenbaum, Hartford, Connecticut.*

Determine the condition which two concentric spheres must satisfy in order that a tetrahedron can be simultaneously inscribed in one and circumscribed about the other. Give a construction for the tetrahedron.

*Solution by the Proposer.* It will be seen that the only requirement is

$$R \geqq 3r, \tag{1}$$

where $R$, $r$ are the radii of the spheres $S$, $s$, having $O$ as their common center. The argument is based on the known theorem that if any pair of the three centers —circumcenter, incenter, centroid—of a tetrahedron coincide, then all three coincide and, further, the four faces are congruent.

Now if $ABCD$ is a tetrahedron satisfying the given requirements, then the circumcircle $C_1$ of the face $ABC$ is the intersection of $S$ with a plane $M$ tangent to $s$. The center $O_1$ of $C_1$ is the point of contact of $M$ and $s$, and its radius is

$$R_1 = \sqrt{R^2 - r^2}. \tag{2}$$

Since $O$ is the centroid of $ABCD$, the line $DO$ pierces the face $ABC$ in its centroid $G$, and

$$DO = 3 \cdot OG. \tag{3}$$

Let the plane $N$ through $D$ parallel to $M$ cut $S$ in the circle $C_2$. The center $O_2$ of $C_2$ is collinear with $O_1$ and $O$, and it follows from (3) that $OO_2 = 3 \cdot O_1O = 3r$, so that, since $OO_2D$ is a right triangle, the radius $O_2D$ of $C_2$ is

$$R_2 = \sqrt{R^2 - (3r)^2}. \tag{4}$$

The similarity of $OO_1G$ and $OO_2D$ gives

$$O_1G = R_2/3, \tag{5}$$

where $R_2$ is given by (4).

A construction for the tetrahedron is now easy. On a circle $C_1$ with radius $R_1$ as given by (2), take an arbitrary point for vertex $C$, and locate any point $G$ satisfying (5). These can always be done if condition (1) is satisfied. Prolong $CG$ half its length to $M$. From the preceding analysis it is easy to show that $CM < 2R_1$, so that $M$ always falls within circle $C_1$. Through $M$ draw the chord $AB$ perpendicular to $O_1M$. Finally locate $D$ as the intersection of $GO$ with $S$. It follows that when condition (1) is satisfied, there exist in general infinitely many noncongruent tetrahedrons inscribed in $S$ and circumscribed about $s$.

Considerable interest attaches to the problem of the set of tetrahedrons belonging to a given pair of circumscribed and inscribed spheres. The special case here treated points to the likelihood that, unlike its analogue in plane geometry the conditions on two spheres that they be the circumscribed and inscribed spheres of a tetrahedron (and perhaps also any polyhedron) is not an equaiity, but rather an inequality—something like $F(R, r, d) > 0$, where $d$ is the distance between the two centers.

**A Step Function Connected with the Law of the Mean**

4371 [1949, 695]. *Proposed by Albert Wilansky, Lehigh University.*

Define $\theta(a, h)$ as the largest number $\theta$ satisfying

(i) $$0 < \theta < 1,$$

(ii) $$f(a + h) = f(a) + hf'(a + \theta h),$$

where $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$.

Now set $\lambda(h) = [h \cdot \theta(0, h)]^{-1}$. Prove that as $h$ tends to zero, $\lambda(h)$ tends to infinity in a step function manner; specifically, given $\epsilon > 0$, there is a number $H(\epsilon)$ such that for every $h$ with $|h| < H$ there is an integer $n(h)$ such that

$$|\lambda(h) - (n + \tfrac{1}{2})\pi| < \epsilon.$$

*Solution by L. A. Ringenberg, Eastern Illinois State College.* With $a = 0$, $h \neq 0$, (ii) becomes

$$h \sin(1/h) = 2\theta h \sin(1/\theta h) - \cos(1/\theta h).$$

Since $h \sin(1/h)$ and $2\theta h \sin(1/\theta h)$ each approach zero with $h$, it follows that $\cos(1/\theta h) = \cos\lambda$ also approaches zero with $h$. Given $\delta > 0$, there is an $H(\delta) > 0$ such that $|h| < H$ implies $|\cos\lambda| < \delta$. Given $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that $|\cos\lambda| < \delta$ implies $|\lambda - (n + \frac{1}{2})\pi| < \epsilon$ for some integer $n$. Thus $|h| < H[\delta(\epsilon)] = H(\epsilon)$ implies

$$|\lambda - (n + \tfrac{1}{2})\pi| < \epsilon$$

for some integer $n$. Since $\lambda = \lambda(h)$ approaches infinity as $h$ approaches zero, it follows that $n$ is a function of $\lambda$, and hence of $h$.

We observe that if $\theta(a, h)$ is any number (not necessarily the largest) satisfying conditions (i) and (ii) then the conclusion of the problem is still valid except that $n$ depends upon $h$ and $\theta$.

Also solved by N. J. Fine and E. G. Kimme.

*Editorial Note.* Fine, by similar analysis, shows that if $f(x)$ is $x^\alpha \sin(1/x^\beta)$, where $x > 0$, $\beta > 0$, and $\alpha$ is an arbitrary real number, then

$$|\lambda(h) - \{(n + \tfrac{1}{2})\pi\}^{1/\beta}| < \epsilon.$$

**Sequence of Sines**

4372 [1949, 695]. *Proposed by Ky Fan, University of Notre Dame.*

For what real values of $x$ does the sequence

$$f_n(x) = \sin 7^n \pi x$$

converge and what is the limit?

*Solution by Robert Steinberg, University of California, Los Angeles.* Consider the more general problem with 7 replaced by any odd positive integer $q \neq 1$, and let angles congruent mod $2\pi$ be considered equivalent. If $f_n(x)$ converges, then

(I) The sequence of angles $s_n = q^n \pi x$ must either have a single limit $t$ (in

the sense of the preceding sentence), or it must separate into two subsequences approaching $t$ and $\pi-t$, respectively. In the former case, $t$ satisfies the relation $qt\equiv t \pmod{2\pi}$ and hence $t=2m\pi/(q-1)$, where $m$ is any integer. In the latter case the subsequences must be $s_{2n-1}\to t$, $s_{2n}\to\pi-t$. (Otherwise it would be possible to select a subsequence of pairs of consecutive terms

$$s_{n_k},\ s_{n_k+1}\to t, \qquad k=1,2,\cdots,$$

whence $t$ would satisfy $qt\equiv t$, and we have only the first case.) Therefore $qt+t\equiv\pi \pmod{2\pi}$ and hence $t=(2m+1)\pi/(q+1)$. In both cases each of the sequences $s_{2n-1}$ and $s_{2n}$ has a single limit $T$ which satisfies the relation $q^2T\equiv T \pmod{2\pi}$.

(II) For some $N$, $s_N\equiv t$; and hence $s_n\equiv t$, $\pi-t$, for $n\geqq N$. For, otherwise, the relation

$$s_{n+2}-t\equiv q^2(s_n-t), \qquad q^2>1, \tag{1}$$

would imply non-convergence for the sequence. Combining (I) and (II) we see that if the sequence $f_n(x)$ converges we have

$$x=\frac{2m}{q^N(q-1)} \quad \text{or} \quad x=\frac{2m+1}{q^N(q+1)},$$

where $m$ and $N$ are any integers, and the limits are respectively

$$\sin\frac{2m\pi}{q-1}, \qquad \sin\frac{(2m+1)\pi}{q+1}.$$

Conversely if $x$ has any of these values, $f_n(x)$ evidently converges to the limit stated.

Also solved by J. B. Kelly, J. G. Millar, L. A. Ringenberg, and the Proposer, and partially solved by P. M. Anselone and Chia-shun Yih.

*Editorial Note.* The above argument applies with little change when $q$ is even. Then, however, the values $x=(2m+1)/q^N(q+1)$ do not satisfy, since

$$s_{n+1}+s_n\equiv 2\pi \pmod{2\pi}, \qquad n>N.$$

Hence, unless $s_n=0$ or $\pi$, $f_{n+1}=-f_n$, contrary to the convergence of $f_n$.

For those who might desire more detail, the argument at (1) is expanded as follows. Let $s_n-t\equiv 2\pi r_n (0\leqq r_n<1)$. Then

$$s_{n+2k}-t\equiv 2\pi(q^2)^k r_n, \qquad k=1,2,\cdots.$$

If $r_n$ is irrational, the residues (mod 1), $r_{n+2k}$ of $(q^2)^k r_n$ are dense in the interval 0, 1, contrary to convergence. If $r_n$ is rational, its denominator $d$ may be taken prime to $q$ (with a larger value of $n$ if necessary). If $d>2$, the residues $r_{n+2k}$ form a finite set of fractions repeated cyclically infinitely often, contrary to convergence. If $d=2$ we have $s_{n+2k}-t\equiv\pi$, which gives non-convergence unless $t=0$ or $\pi$. If $d=1$, then $s_{n+2k}=t$ for all $k$.

## RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116 Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.*

*Primer of College Mathematics.* By J. F. Randolph. New York. The Macmillan Company, 1950. 13+545 pages. $4.75.

This book is an interesting and well constructed text, readable, rigorous, discriminating in choice of material, and refreshing in manner of presentation. The format is good, the appearance of the pages uncrowded and attractive.

In the preface the author says, "This book is written to try to restore some of the unity to elementary mathematics." Trigonometry, college algebra, and analytical geometry are well blended throughout. Application is made at appropriate points to simple concepts of investment and statistics. An optional chapter is included on the calculus applied to algebraic and logarithmic functions. Intended for a freshman course, the "Primer" would provide training both thorough and efficient, since the author has seized the advantage of a unified approach by presentation of ideas just when the student is most receptive to them. The ample content and close integration of subject matter imply to the reviewer that the text would probably be used more successfully in a ten-hour than in a six-hour course.

The book opens with permutations and combinations. These lead to a study of integers, and this in turn to further review of topics in algebra. The student thus gets off to an interesting and absorbing start learning new material and recalling old. Throughout the text each new concept is introduced by use of a specific problem skillfully chosen and presented so that the idea which the author wishes to develop stands out clearly. Similarly, definitions and symbols are introduced in so natural a manner and as filling such an obvious need that students will use them without realizing that they are new and therefore probably hard. As important concepts recur in later parts of the text, their significance and power become more apparent. For instance, the introduction of curve plotting in a paragraph entitled *Sets of Ordered Pairs and Sets of Points* emphasizes the functional aspect of a graph and also makes use of parametric representation quite natural, although equations of curves have not yet been mentioned and formal work in graphical analysis also comes later.

This "spiral" approach is correct pedagogically when in the hands of as competent a writer as Professor Randolph. He has made sure that important concepts recur and that problems and exercises give continual cumulative review. By use of this text the student should augment his knowledge by a very sound, interesting, and satisfying process. The instructor should enjoy teaching it, for it abounds with new and fruitful methods of dealing with traditional material.

JULIA W. BOWER

*Modern College Geometry*. By D. R. Davis. Cambridge. Addison-Wesley Press, Inc., 1949. 9+238 pages. $4.50.

This book seems to strike a happy medium between over-emphasis upon construction as one extreme and classical theorem study as the other. The first six chapters concerning geometric procedures, loci, fundamental theorems, similar figures, auxiliary figures, and harmonic ranges should open up a new world to the average non-mathematical student who later might find himself trying to teach high school geometry.

The last four chapters bring to light many of the modern discoveries concerning the triangle and the circle. While the average reader may not remember for long the distinction between a Steiner point and a Tarry point, or the relation between a Miguel triangle and a Simpson line, yet his vision is enlarged by knowing where to find such interesting modern geometric concepts.

The author of this book has wisely included many of the essential high school plane geometry theorems in the first chapter. This feature is a welcome oasis for both teacher and student of college geometry.

A reasonable number of problems are available in each chapter. The bibliography of twenty-four books listed on page 233 should preclude the reader from believing that geometry is static or died with Euclid.

The author and publisher have rendered a valuable service to present day students and teachers of geometry.

W. R. HUTCHERSON

*First Course in Probability and Statistics*, Vol. I. By J. Neyman. New York. Henry Holt and Company, 1950. 9+350 pages. $3.50.

According to the author, this book is intended to provide material for a beginning one-semester course. Three different categories of students are contemplated: (1) students who would like to take just one course in mathematical statistics for purposes of general education (2) prospective future mathematical statisticians and (3) students who specialize or intend to specialize in one of the fields of application.

The purpose of this basic course is to introduce the most fundamental concepts of modern statistical theory and to connect these concepts with as many fields of application as is practicable. It is assumed that the student's mathematical education is limited to high school algebra.

It has long been felt by many statisticians that the basic ideas should be taught at the beginning of any sequence of courses in statistics, but some have expressed doubt that these could be made comprehensible to students with a very limited mathematical background. Because the author has been, and is, so intimately connected with the development of modern statistical theory as well as being a successful teacher in that field, he is eminently qualified to undertake this important task.

There are five chapters in this volume. The first chapter is an introduction and attempts to explain the scope of statistics. Chapter 2 introduces the

basic concepts of probability and the rules concerning the combinations of probabilities. The third chapter is concerned with applications of the theory of probability to genetics. In the fourth chapter the ideas of random variables and frequency distributions are discussed. The final chapter is the capstone of the book. In this chapter the author considers the theory of testing hypotheses and examines the ideas involved (errors of the first and second kind, power of a test, best critical regions, etc.) giving numerous examples to bring out the delicate points involved in these concepts. This is carefully done and will prove very valuable to the conscientious student.

Since this book is written for different classes of students there are starred sections which may be omitted with those students who do not have the mathematical preparation. Some ideas from the calculus are used and for those students who have no acquaintance with that discipline, the instructor will need to elaborate the passages in the book.

There are a number of interesting and novel problems phrased so as to stimulate the student's critical thinking. Emphasis is placed on careful and exact statements, which is so often lacking in many of the "practical" statistics texts.

There are several references given at the ends of Chapters 1, 3 and 5, some of these being fundamental research papers which will prove difficult reading for most of the students to whom this book is addressed. No reference is given at the end of Chapter 2, although there must be some that the student could read profitably.

With the help of a competent and well-trained instructor, this book can be most useful in introducing many basic concepts of statistics to students. For this purpose it is one of the best, if not the best, that has yet been written.

J. R. VATNSDAL

*Introduction to the Theory of Statistics*. By A. M. Mood. New York. McGraw-Hill, 1950. 9+433 pages. $5.00.

Although mathematical statistics has made great strides in the last twenty-five years, the subject remains, at least in part, an oral one. The principal media for initial orientation remain the lectures and notes of lectures in the relatively few schools where the modern theory is being developed. The result is that students without access to these sources are likely to be misled as to the nature of statistics and the problems to be investigated. Often competent mathematicians who attempt to work in statistics spend their time on unimportant problems.

The situation is little better when it is a question of instructing the people who will apply the theory to practical problems. The books in wide use are "cookbooks," that is, books which lay down dicta and describe rules, without explaining their logic, or the assumptions underlying the rules, or the precise consequences of the rules. The student whose knowledge comes exclusively from such books is at a loss when he encounters a problem not treated in these books

(he often cannot recognize that the problem has not been treated in his textbooks). These books describe at great length good methods of computing statistics and then devote little space to telling the student how to answer the questions which gave rise to the computations. A famous book discusses numerous tests of significance (tests of hypotheses) without once telling the reader what such tests are, the meaning of "null hypothesis," *etc.*

This book by Mood with its modest title is an excellent introduction to statistics. At present writing (May, 1950) only one other such book is known to the reviewer. Mood's book presupposes a knowledge of the calculus, but no prior knowledge of probability or statistics is required; indeed, the first part of the book is devoted to an introduction to probability. Numerous interesting problems carry the theory further, and are a challenge to the student and a blessing to the instructor. Among the chapter headings we may mention the following: point estimation, interval estimation, tests of hypotheses, regression and linear hypotheses, experimental designs and the analysis of variance, sequential tests of hypotheses.

The book, of course, has a few inadequacies. The description of a test of hypothesis on page 245 is very meager and inadequate. The central limit theorem is considerably more than the very special case described on page 136. Formula 6 on page 266 is incorrect. So is the last sentence in the first paragraph on page 157. The reviewer certainly does not agree that the whole problem of point estimation is virtually solved (page 161), nor would he agree with all other value judgments of the author. The reviewer regrets that the author has nowhere described the fact that many important problems in statistics are multi-decision problems for which the classical theory is inadequate. Such a discussion could well have accompanied a more adequate exposition of the notion of testing hypotheses. A few words about a modern approach to these problems and references to other literature would have appealed to the good student.

Some of the above criticisms represent divergences of opinion between the author and the reviewer, on which the informed reader must have his own opinion. Other criticisms refer to errors from which no book, no matter how excellent, can be free, and which could be corrected in subsequent editions. The author is to be highly complimented for a job well and painstakingly done.

JACOB WOLFOWITZ

## NEW BOOKS RECEIVED

*A Manual for the Slide Rule.* By P. E. Machovina. New York, McGraw-Hill, 1950. 73 pp. $.75.

*Wo steckt der Fehler?* By W. Lietzmann. Leipzig, B. G. Teubner, 1950. 183 pp. No price.

*Einführung in die Analytische Geometrie.* By L. Bieberbach. Bielefeld, Verlag für Wissenschaft und Fachbuch G. M. B. H., 1950. 168 pp. No price.

*Analytic Geometry and Calculus.* By L. M. Kells. New York, Prentice-Hall, Inc., 1950. viii+623 pp. $4.75.

*Corso di Matematica*. Algebra—Parte Prima. By G. Andruetto and A. Corio. Turin, Italy, G. B. Paravia and Co. 1950. No price.

*Operational Calculus*. By B. Van der Pol and H. Bremmer. New York, Cambridge University Press, 1950. xiii+415 pp. $10.00.

*Technological Applications of Statistics*. By L. H. C. Tippett. New York, John Wiley and Sons, Inc., 1950. ix+189 pp. $3.50.

*Algebraic Technique of Integration*. By H. F. MacNeish. Florida, University of Miami Press. viii+109 pp. No price.

*Elasticity*, Vol. III, Proceedings of Symposia on Applied Mathematics. By Churchill, Reissner and Taub. New York, McGraw-Hill Book Co., 1950. v+233 pp. $6.00.

*Fourier Transforms* (Annals of Mathematics Studies, No. 19). By S. Bochner and Chandrasekharan. Princeton University Press, 1949. 219 pp. $3.50.

*The Nomogram*. By H. J. Allcock and J. Reginald Jones. New York, Pitman Publishing Corp., 1950. x+238 pp. $3.75.

*Differential and Integral Calculus*. By Edmund Landau. New York, Chelsea Publishing Company, 1950. 366 pp. No price.

*Elementary Theory of Equations*. By Samuel Borofsky. New York, The MacMillan Co., 1950. x+302 pp. $4.25.

*Mathematical Engineering Analysis*. By Rufus Oldenburger. New York, The MacMillan Co., 1950. xiv+426 pp. $6.00.

*Principles of Finance and Investment*. By L. G. Whyte. Cambridge, England, Cambridge University Press, 1950. 8+176 pp. $2.50.

*Introduction to Linear Algebra and the Theory of Matrices*. By Hans Schwerdtfeger. Holland, P. Noordhoff, 1950. 280 pp. 17½ Guilders.

*Teirica dell'ammortamento*. By Insolera, Filadelfo. Turin, Italy, Giulio Einaudi, 1950. 304 pp. Paper bound, Lire 2400.

*Introduction to Mathematical Logic and Its Applications*. By Ira Rosenbaum. Florida, University of Miami Press, 1950. 98 pp. No price.

*Einführung in die Höhere Algebra*. By Gunter Pickert. Gottingen, Vandenhoeck and Ruprecht, 1951. 298 pp. DM 14.80.

*Mathematics—Queen and Servant of Science*. By E. T. Bell. New York, McGraw-Hill, 1951. xx+437 pp. $5.00.

*Life Insurance and Mathematics*. By R. E. Larson and Erwin A. Gaumnitz. New York, John Wiley and Sons. vii+184 pp. $3.75.

*The Hodograph Method in Gas Dynamics*. By Dr. A. G. Ghaffari. Tahran Taban Press, 1950. iv+129 pp. No price.

*College Algebra*. By H. K. Fulmer and Walter Reynolds. Boston, Ginn and Company, 1951. 204+xiv pp. $2.85.

*Demography*. By P. R. Cox. New York, Cambridge University Press, 1951. xii+326 pp. $4.00.

*Statistics*, Vol. II. By N. L. Johnson and H. Tetley. New York, Cambridge University Press, 1951. xii+318 pp. $4.00.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

### CLUB REPORTS, 1949–50

EDITORIAL NOTE. Letters have been sent to the Heads of Mathematics Departments at various colleges and universities, requesting that the Sponsor of the Mathematics Club, Society, or Fraternity at that institution record the activities of his organization with the Editor of this department of the MONTHLY. The reports should be sent in immediately, as the publication date will depend on the order in which the records have been received. All mathematics organizations are invited to submit a neatly written summary of their activities, even though no letter has been received. When writing up the activities, please follow the form which is used on these pages.

#### Mathematics Society, St. John's College

The *Mathematics Society* of St. John's College reports the following activities for 1949–50:

Professor Tolle of St. John's University Graduate School spoke on *Transfinite numbers* at one of the meetings.

The society publishes *The Mathazine*, a magazine for undergraduates, to help publicize the mathematics department and the society. A guidance clinic, to aid those who have not yet decided upon a career, was held at which speakers discussed the various possibilities open to mathematicians and the necessary prerequisites for each. The society made a field trip to Brookhaven National Laboratories at Upton, New York.

The year's activities were climaxed by a banquet at which an award was made to Gennaro Montanino, the winner of the Mathematics Contest in the college. Certificates were also awarded to members elected to the Honors Council, a group of outstanding mathematics students.

Officers elected for the year 1950–51 were: President, Carl Wampole; Secretary, Richard Ziesig.

#### Mathematics Club, University of Dayton

The *Mathematics Club* of the University of Dayton met bi-weekly throughout the school year 1949–50. The papers presented at these meetings were:

*An introduction to group theory*, by Joseph Schell
*Asymptotes to curves of infinite extent*, by Eduardo Mulanovich
*The phase rule*, by Jorge Nunez
*Polynomial divisibility*, by Demetriuz Zonars
*The sign of the quadratic polynomial*, by Aspasia Zonars
*Physical interpretation of concepts in vector analysis*, by Richard Segers

*Numerical solution of Laplace's equation as applied to photoelasticity*, by Joseph Gallagher

*Servomechanism analysis*, by Robert A. Thomson, S.M.

*Matrix algebra*, by T. David Riney

*Bessel functions and their application to cylindrical wave guides*, by Albert Windeljohaun

*Evaluation of numbers*, by Dr. K. C. Schraut.

The club held a joint meeting with the *Chemistry Society* during the winter season at which Dr. Burbage of the Monsanto Chemical Company lectured on *The use of radio-active materials*.

In the spring, several members of the club accompanied Dr. Schraut to the University of Michigan for an undergraduate conference on Mathematics in conjunction with the Michigan Section of the Association. At this meeting the following two papers were presented by members of our group:

*The inverse integral of the Laplace transformation*, by Christof Neugebauer

*The complex fourier series*, by Richard Segers.

Demetrius Zonars was presented the Dean of Science award at the annual dinner for the best paper of the first semester and Richard G. Segers received the award for the second semester.

Prof. M. O. Thurston, Acting Head of the Department of Physics of the Air Force Institute of Technology, spoke on *Oscillations, classic and atomic* at the final meeting of the year.

The officers for 1949–50 were: President, Joseph Schell; Vice-President, Gordon Mills; Recording Secretary, T. David Riney; Publicity Secretary, Bro. R. A. Thomson, S.M.; Treasurer, Hilary Allemeier; Faculty Advisor, Dr. K. C. Schraut.

**James G. White Mathematics Club, University of Kentucky**

The *James G. White Mathematics Club* is open to anyone interested in mathematics and held seven meetings during the 1949–50 academic year. The following programs were presented:

*Some applications of Archimedes theories*, by D. C. Rose

*A few remarks on* $\tan n\theta$, by M. C. Brown

*Committee report on a six months' survey of the occupational opportunities for Mathematics majors*, by Allen Wilson, Ernest Steele, Genevieve Snider, and John Wells

*Mathematical curiosities*, by Dr. H. H. Downing

*Matrices*, by Dr. J. A. Ward.

Officers serving for the year 1949–50 were: President, Lillian Griffey; Secretary-Treasurer, Ann Wiesman; Faculty Sponsor, Ruric E. Wheeler.

**Order of Magnitude, Pasadena City College**

The *Mathematical Section* of the *Order of Magnitude*, an organization made up of mathematics students and astronomy students, at Pasadena City College held the following programs during 1949–50:

*Magic squares*, by Marvin Ridley
*Apollonius' problem*, by Robert Phelan
*Angle trisections*, by Ray Kilgrove
*Binary numbers*, by Marvin Ridley.

The club meets every two weeks, at which times a member of the club or a guest speaker gives a talk on some interesting mathematical topic or a related subject. The Mathematical Section and the Astronomy Section had one joint meeting during the year.

Social activities included a Christmas Party and a trip to the Calico Mountains.

Officers for the first semester were: President, Marvin Ridley; Vice-President, Andy Markell; Secretary-Treasurer, Robert Phelan.

Officers for the second semester were: President, Robert Phelan; Vice-President, Andy Markell; Secretary-Treasurer, Nancy O'Dell.

**Pi Mu Epsilon, Hunter College**

During 1949–50, the *New York Beta* chapter of *Pi Mu Epsilon*, continued its practice of presenting papers at each meeting.

The topic for the Fall semester was *Vector analysis*, and papers were presented by: Ruth Grabenheimer, Therese Schwarz, Shirley Schneiderman, Mary Shanahan, Yvonne Roach, Doris Cohen, Robert Sanders, Edith Bruton, Goria Bullock, Joan Salatino, Beverly Schiff, and Fay Gorenstein.

The topic for the Spring semester was *Algebraic Geometry*, with the following participating: Doris Matthews, Eloise Solomon, Carla Schmitz, Pearl Goldstein, Hermia Hochberg, Seymour Goldberg, Carol Weisner, Marie Roston, Flora Johns, and Hanna Fischer.

The main event of the year was the 25th Anniversary Dinner held in April. Dr. Tomlinson Fort, Former Chairman of the Department of Mathematics at Hunter College at the time of the founding of the New York Beta chapter of Pi Mu Epsilon, addressed the group. He urged all the women present to take an active part in the mathematical research being conducted at the present time.

Dr. H. M. MacNeille, Executive Director of the American Mathematical Society, spoke on *The International Congress of Mathematics*. Professor Bushey introduced some outstanding alumnae who spoke to the members about their experiences and the work they are engaged in.

The following officers were elected for 1950–51: President, Marie Barbiere; Corresponding Secretary, Ursel Kramer; Recording Secretary, Dorothy Freudenberger; Treasurer, Hannah Stein.

## NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.*

### INSTITUTE FOR TEACHERS OF MATHEMATICS

The University of California at Los Angeles announces that an Institute for Teachers of Mathematics will be held on its campus, July 9–20, 1951. Professor W. W. Rankin of Duke University will be the Director of the Institute.

Two lectures daily will be given by outstanding men from business, industry, and teaching. The following study groups will be conducted daily by leaders of recognized ability chosen from teachers in the various fields of mathematical instruction:

1. Aids in the teaching of geometry and algebra.
2. The development and coordination of basic ideas in mathematics.
3. Understanding in the teaching of arithmetic.
4. The teaching of applications of mathematics used in industry and other fields.
5. Use of mathematical instruments.
6. Modern trends in the organization of secondary curricula in mathematics.

Certificates of attendance will be issued to those registered in the Institute. It will also be possible for registrants to earn two units of college credit by attending lectures and by participating in a sufficient number of study groups.

Application blanks for the Institute and further details may be obtained by writing to Professor Clifford Bell, Mathematics Department, University of California, Los Angeles 24, California.

### ANNUAL SUMMER MEETING OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The National Council of Teachers of Mathematics is holding its annual Summer Meeting at St. Olaf College, Northfield, Minnesota, on August 20–23, 1951. Outstanding teachers and nationally known leaders in the field of mathematics will conduct meetings and present demonstrations at all levels of mathematics. The meetings will include general sessions, sectional meetings, laboratory sessions, discussion periods, and study groups. These study groups will be under the direction of prominent teachers and leaders for a continuing three-day session and will have limited enrollments. The latest in audio-visual aids will be exhibited; films, film-strips, models. Anyone having a model that has proven useful in the teaching of mathematics is asked to exhibit the model at this con ference. Write to Mr. Emil Berger, Monroe High School, St. Paul, Minnesota, for information if you wish to exhibit material.

The college dormitories will provide adequate accommodations; meals may be obtained at the college cafeteria. Reservations for rooms and registration

for study groups should be made as soon as possible. Send reservations or requests for information and registration forms to Professor C. S. Carlson, St. Olaf College, Northfield, Minnesota. A registration and reservation form will appear in the May issue of *The Mathematics Teacher*.

### CARE-UNESCO BOOK FUND

The CARE-UNESCO Book Fund is a program which supplies important new technical books to universities, libraries and medical and scientific centers overseas. Its primary purpose is to bring as many professional people as possible up-to-date on current developments in their fields. The Fund has been established in cooperation with UNESCO, the Library of Congress, the American Library Association, medical and scientific groups, and governmental authorities overseas.

By making a general contribution, in any amount, you can share directly in this important program. Your gift will be allocated on the basis of lists compiled by the Fund in consultation with UNESCO and ministries of education. Individuals and organizations giving $10 or more may request books for a specific institution or indicate their choice of country, type of institution and category of books.

Further information may be secured by writing to the CARE-UNESCO Book Fund, 20 Broad Street, New York 5, or your local CARE Office. The Fund will be glad to advise as to where books are most needed and to provide interested groups with additional material, posters and films.

### SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1951:

*Columbia University, Teachers College.* July 2 to August 10: Mr. Campbell, professionalized subject matter in junior high school mathematics, current problems in teaching arithmetic; Professor Carnahan, teaching algebra in secondary schools, teaching geometry in secondary schools; Professor Clark, teaching arithmetic in the elementary school; Professor Fehr, current problems in teaching secondary school mathematics, professionalized subject matter in advanced secondary school mathematics; Professor Shuster, business mathematics, field work in mathematics; Professor Yates, applications of plane geometry, teaching of elementary college mathematics.

*De Paul University.* June 25 to July 31: Professor Caton, probability, mathematical foundations of statistics; Professor De Cicco, infinite series, finite differences.

*Oklahoma Agricultural and Mechanical College.* June 18 to July 14: Professor Ahlfors of Harvard University, conformal mapping. June 18 to July 2: Professor Tarski of University of California at Berkeley, arithmetical classes and types of mathematical systems.

*University of California at Berkeley.* June 18 to July 28: Professor Davis of Northwestern University, non-linear problems.

*University of Tennessee.* June 11 to August 24: Staff, ordinary differential equations, theory of equations, modern geometry; Professor Harrold, topology; Professor Snyder, theory of measure (given at Oak Ridge Institute of Nuclear Studies); Professor Wilson, theory of matrices, complex variable (given at Oak Ridge Institute of Nuclear Studies).

## PERSONAL ITEMS

Professor W. W. S. Claytor of Howard University was the representative of the Association at the celebration of the One Hundredth Anniversary of Miner Teachers College, Washington, D. C. on March 4–10, 1951.

Professor P. A. Caris of the University of Pennsylvania and Professor C. O. Oakley of Haverford College were the delegates of the Association at the Fifty-fifth Annual Meeting of the American Academy of Political and Social Science, which was held in Philadelphia, Pennsylvania, April 6–7, 1951.

Arkansas Polytechnic College announces that as a result of its transition from a junior college to a four year college conferring degrees academic rank has been granted to its teaching staff as follows: Professor J. R. Abernithy, Associate Professor Maggie B. Davis, Assistant Professor E. H. Ham, Assistant Professor Aleze G. Fullerton, Assistant Professor Maude Moore.

De Paul University announces the promotions of Assistant Professor W. B. Caton to an associate professorship and Lecturer E. P. Merkes to an instructorship.

Trinity College, Hartford, Connecticut, reports that Dr. E. N. Nilson of Trinity College and Mr. S. L. Crossman and Mr. Walter Ramshaw of the United Aircraft Computing Laboratory have developed a course which combines numerical mathematical analysis with the use of IBM punch card computing machinery. Lectures on numerical analysis and machine methods are given at the College, supplemented by a laboratory period at the United Aircraft Computing Laboratory where students work with the latest types of IBM electronic computing equipment.

University of Alberta makes the following announcements: Associate Professor E. S. Keeping has been promoted to a professorship; Assistant Professor Max Wyman has been promoted to an associate professorship; Dr. D. R. Crosby has been appointed to an assistant professorship; Mr. T. M. Fostvedt, Mr. R. C. Jacka, Mr. A. Shaw and Mr. L. A. Fisher have been appointed Lecturers.

Miss Florence R. Anderson of the University of Southern California has accepted a position as research laboratory analyst with Northrop Aircraft, Incorporated.

Assistant Professor Lulu Bechtolsheim of the University of Redlands has been promoted to an associate professorship.

Mr. Jonas Beraru, formerly a research analyst at North American Aviation, Incorporated, is now a mathematician at the Rand Corporation.

Miss Mary P. Burkart, previously a graduate assistant at the University of Detroit, has been appointed to an instructorship at Trinity College, Washington, D. C.

Professor Emeritus W. B. Carver of Cornell University has been teaching at the University during the second semester of the current academic year.

Mr. E. J. Cogan of Pennsylvania State College at Pottsville has been transferred to Pennsylvania State College at State College, where he has a position as part-time instructor.

Mr. K. L. Cooke, previously a graduate student at Stanford University, has been appointed to an instructorship at State College of Washington.

Mr. G. A. Culpepper of the University of Colorado has accepted a position as mathematician at White Sands Proving Ground.

Mrs. Alice B. Dickinson, who was a teaching fellow at the University of Michigan, has been appointed to an instructorship at Pennsylvania State College.

Dr. O. L. Dustheimer has been appointed Pension Consultant for D. Miley Phipps and Associates, Cleveland, Ohio.

Miss Ruth B. Eddy is teaching at Hope High School, Providence, Rhode Island.

Mr. R. L. Eisenman of the University of Connecticut has been appointed to an instructorship at the University of Maryland.

Mr. W. E. Felling, graduate student at St. Louis University, has been appointed to an instructorship at Parks College.

Mr. F. H. Fisher, previously a teaching fellow at West Virginia University, has a position as teacher at McDonald High School, McDonald, Pennsylvania.

Mr. A. G. Hansen of the University of Maryland has accepted a position as aeronautical research scientist for the National Advisory Committee for Aeronautics, Cleveland, Ohio.

Mr. J. J. Hart, previously a graduate student at the University of Alabama, has been appointed to an instructorship at Tennessee Polytechnic Institute.

Dr. R. T. Hood of Beloit College is now Pastor of the Lincoln Baptist Church, Macedon, New York.

Professor L. A. Hopkins of the University of Michigan has retired with the title of Professor Emeritus.

Dr. C. C. Hsiung of the University of Wisconsin has been appointed Lecturer at Northwestern University.

Assistant Professor Fritz John of New York University has a position as mathematician at the Institute for Numerical Analysis.

Mr. L. G. Jones, formerly a graduate assistant at the University of Oregon, has been appointed to the position of mathematician in the Physical Research Unit, Boeing Airplane Company, Seattle, Washington.

Assistant Professor Wilfred Kaplan of the University of Michigan has been promoted to an associate professorship.

Dr. Samuel Karlin of California Institute of Technology has been appointed to an assistant professorship at Princeton University.

Mr. C. E. Kelley of the University of Missouri has accepted a position as teacher in the public school system of Texarkana, Texas.

Dr. P. J. Kelly of the University of Southern California has been appointed to an associate professorship at the University of California, Santa Barbara.

Mr. A. L. Lanckton of Socony Vacuum Oil Company, Athens, Greece, has been transferred to Istanbul, Turkey.

Mr. J. N. P. Lawrence, previously a student at Johns Hopkins University, has an A.E.C. Radiological Physics Fellowship at Vanderbilt University.

Mr. Joseph Levitt of Pratt Institute has been appointed Process Engineer at the M. W. Kellogg Company, Jersey City, New Jersey.

Mr. R. W. Long of New York University has been appointed to an assistant professorship at Washington and Jefferson College.

Mr. W. C. Lowry of Kent State University is now a part-time instructor at Ohio State University.

Mr. Edward McGaughy of Lawrence College has been appointed Lecturer at Columbia University.

Professor B. E. Mitchell, Millsaps College, has been appointed to a professorship at the University of Mississippi.

Mr. Benjamin Mittman, previously a student at Illinois Institute of Technology, has been appointed Teaching Assistant at the University of California at Los Angeles.

Mr. D. S. Park has accepted a position as mathematician at the National Bureau of Standards.

Mr. M. O. Peach, Carnegie Institute of Technology, has been appointed to an associate professorship at the University of Notre Dame.

Assistant Professor C. L. Perry, Jr., University of Arkansas, has a position as mathematician at the Oak Ridge National Laboratory.

Dr. G. W. Petrie, III, has been appointed Government Representative for the International Business Machine Corporation, Washington, D. C.

Associate Professor L. V. Robinson of the University of South Carolina has a position as mathematician at the Aberdeen Proving Ground, Maryland.

Mr. W. G. Rouleau of George Washington University has accepted a position as mathematician at the Army Map Service, Washington, D. C.

Mr. J. W. Sawyer of the University of Missouri has been appointed to an assistant professorship at the University of Georgia, Atlanta Division.

Mr. O. T. Schultz, formerly assistant section head of research at Curtiss-Wright Corporation, Ohio, has been appointed to a position as physicist at the Naval Ordnance Laboratory, Silver Spring, Maryland.

Assistant Professor James Singer of Brooklyn College has been promoted to an associate professorship.

Assistant Professor D. L. Thomsen, Jr., Haverford College, has received an appointment as research fellow at California Institute of Technology.

Mr. P. M. Treuenfels has a position as mathematician in the Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland.

Assistant Professor R. S. Britton of New York University died on February 2, 1951.

Professor Emeritus Daniel Buchanan of the University of British Columbia died on December 1, 1950. He was a charter member of the Association.

Associate Professor Emeritus Otto Dunkel of Washington University died on January 16, 1951. He was a charter member of the Association. A more detailed statement will be given in a forthcoming issue of this journal.

Professor Emeritus N. B. Heller of Temple University died on January 31, 1951.

Assistant Professor J. G. Millar of the Calgary Branch of the University of Alberta died on December 9, 1950.

Professor Emeritus G. A. Miller of the University of Illinois died on February 10, 1951. He was a charter member and honorary life member of the Association. A more detailed statement will be given in a forthcoming issue of this journal.

Professor J. F. Ritt of Columbia University died on January 5, 1951. He had been a member of the Association for twenty-eight years.

Professor Emeritus W. H. Roever of Washington University died on January 31, 1951. He was a charter member of the Association.

Professor Abraham Wald of Columbia University died on December 13, 1950.

Professor Emeritus F. S. Woods of Massachusetts Institute of Technology died on December 1, 1950. He was a charter member of the Association.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held at the University of Minnesota, Duluth Branch, in Duluth, Minnesota, on Saturday, October 7, 1950. Sessions were held in the forenoon, at luncheon, and in the afternoon. Professors W. R. McEwen, F. C. Smith and H. M. Anderson presided at the respective sessions.

Forty-three persons attended the meeting, including the following twenty-seven members of the Association: H. M. Anderson, F. J. Arena, J. E. Bearman, L. E. Bush, H. D. Colson, J. C. Cothran, C. B. Germain, Ruby M. Grimes, A. G. Hill, J. S. Hill, J. E. Hafstrom, W. C. Kalinowski, Karlis Kaufmanis, W. H. McBride, W. R. McEwen, A. G. Montgomery, M. J. Norris, L. W. Sheridan, S. C. Simonson, Sister M. Mercedes, Sister M. Prudentia, F. C. Smith, R. C. Staley, Irwin Stoner, A. G. Swanson, Takaski Terami, Matilda B. Thompson.

The committee on high school contests reported that the committee on contests of the High School Principals' Association of Minnesota had not approved the application of the Minnesota Section to hold a mathematics contest in the high schools of the State, and that the committee felt that without the coopera-

tion of the high school officials it would be futile to make any further attempt to hold a mathematics contest.

By invitation of the executive committee, Professor R. E. Graves of the University of Minnesota delivered an address at the morning sessio ı. The title of his address was *Intuitionism versus Formalism*. Abstract of this address follows:

As a science develops it frequently has occasion to re-examine its own foundations, to modify them, or, more rarely, to discard these foundations entirely and to develop a new set in order better to meet certain objectives or subjective criteria of that science. To a large extent, such a critical analysis of foundations may appear to belong to the domain of philosophy, or even psychology, rather than to the specific science under consideration. However, the results of such analysis may have a profound and far-reaching influence upon the entire structure of the science. From these remarks, it should be evident that the borderline between a science and the philosophy of that science may be of a somewhat dubious character. Perhaps as good a way as any to make the distinction is to say that a scientist makes investigations *in* his science, while the philosopher of the science makes investigations *about* the science. Of course, on the basis of such a definition there are many who must be classed as both scientist and philosopher. After these introductory remarks, the speaker mentioned the various schools of thought concerning the foundations of mathematics. The remainder of the address was devoted to a comparison of the main features of the *intuitionism* of Brouwer and the *formalism* of Hilbert. In this comparison, particular emphasis was placed upon the radically different roles assigned to language (natural or formal) in the two doctrines.

The following five short papers were presented:

1. *Recurrence relations for the n-th derivative of the circular functions*, by Professor F. J. Arena, North Dakota State College.

Often in certain Taylor series expansions the successive derivatives of the circular functions are needed. The work of calculating the successive derivatives of some of the circular functions becomes rather complicated. In order to simplify this task the speaker gave recurrence relations for the $n$-th derivative of the circular functions, and then calculated the first six derivatives of tan $(ax+b)$ and sec $(ax+b)$.

2. *Alternative methods of treating increasing and decreasing annuities*, by Professor F. C. Smith, College of St. Thomas.

In most elementary texts in the mathematics of finance, the formula for the amount of an increasing annuity and the formula for the present value of a decreasing annuity are developed, but no direct methods are given for calculating the present value of an increasing annuity or the amount of a decreasing annuity. In this paper, the author proves that if one takes the increment negative, the formula for the amount of an increasing annuity will give the amount of a decreasing annuity; and if one takes the decrement negative, the formula for the present value of a decreasing annuity will give the present value of an increasing annuity.

3. *A slide rule for steam power plant loading calculations*, by Professor A. G. Montgomery, College of St. Thomas and Northern States Power Company.

Steam-electric generating stations in an interconnected system are assigned their power load so that the derivatives of their total cost-vs-power functions are equal. The calculations previously required were outlined in the case of three stations whose cost functions were inter-related by transmission line losses. It was then shown how this procedure—requiring construction of numerous graphs with various multiplications and divisions with values obtained therefrom—was simplified and greatly shortened by use of a special slide rule. It was noted that the calculations are readily adaptable to the use of amplifiers and associated servo-mechanisms but initial cost may prove prohibitive.

4. *A note on W-surfaces*, by Mr. Dale Woods, North Dakota State College (introduced by Professor A. G. Hill).

After a discussion of surfaces of centers, general $W$-surfaces and special $W$-surfaces, the author made use of a theorem about convex $W$-surfaces to derive some interesting theorems about the euclidean sphere.

5. *Some necessary convergence conditions*, by Professor M. J. Norris, College of St. Thomas.

The obvious generalization, and the analog for integrals, of the following result is proved. If $\sum a_n$ is a convergent series of positive terms and $na_n$ is decreasing, then $a_n n \log n$ approaches zero.

L. E. BUSH, *Secretary*

## NOVEMBER MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Oklahoma City on Monday, November 13, 1950. Professor J. T. Krattiger, Chairman of the Section, presided.

Sixty-six persons attended the meeting, including the following thirty-five members of the Association: E. F. Allen, R. V. Andree, Jean M. Baldwin, Arthur Bernhart, J. C. Brixey, N. A. Court, R. B. Deal, A. H. Diamond, R. C. Dragoo, N. A. Eisen, A. A. Grau, E. V. Greer, L. D. Gregory, Claire A. Harrison, J. O. Hassler, W. N. Huff, S. L. Hull, H. V. Huneke, P. W. M. John, J. T. Krattiger, J. E. LaFon, H. I. Lane, H. W. Linscheid, R. D. McDole, G. E. Meador, R. D. Morrison, R. R. Murphy, C. M. Pirrong, D. P. Richardson, W. E. Roth, Harold Shniad, D. R. Shreve, T. C. Smith, C. E. Springer, J. H. Zant.

At the business session the following officers were elected: Chairman, C. M. Pirrong, Jr., Oklahoma City University; Vice-Chairman, R. R. Murphy, Panhandle A. and M. College; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following seven papers.

1. *The Laplace transform in a boundary value problem*, by Professor S. L. Hull, University of Arkansas.

Attention was centered on the use of the Laplace transform in solving certain partial differential equations.

2. *A new method of computing matric inverses*, by Professor R. V. Andree, University of Oklahoma.

This paper was published in this MONTHLY, vol. 58, pp. 87–92.

3. *The solution of equations in matrices*, by Professor W. E. Roth, University of Tulsa.

A solution, $X$, of the unilateral equation

$$A(X) = A_0X^m + A_1X^{m-1} + \cdots + A_m = 0,$$

where the $A_i$, $i=0, 1, 2, \cdots, m$, are $n \times n$ matrices, must be such that its characteristic function, $\phi(\lambda)$, is a divisor of the determinant $|A(\lambda)|$. This fact makes it possible to determine a second equation, $\phi(X)=0$, which may be solved simultaneously with the given equation, $A(X)=0$, by

eliminating all powers of $X$ higher than the first between them. This procedure was discussed and examples showing its application and limitations were given.

4. *A simple postulational development of the real numbers*, by Mr. R. B. Deal, University of Oklahoma.

Postulates and definitions for the real numbers were given. The positive integers were characterized as a well ordered set $S$ such that: (1) $S$ has no last; (2) every bounded set has a last. This system was proved to be categorical.

5. *A comparison of the classical notions of completeness for the real numbers*, by Professor Casper Goffman, University of Oklahoma, introduced by Professor J. C. Brixey.

This was an expository report on the relation between completeness in the sense of Archimedes and in the Cantor Dedekind sense for ordered abelian groups.

6. *Loxodromes on a general surface*, by Professor C. E. Springer, University of Oklahoma.

In this paper the differential equations of the curves on a general surface which make a constant angle with the curves along which the curvilinear coordinate $u^1$=constant were obtained. These equations were compared with the differential equations of the union curves with respect to a congruence of lines referred to the surface. A unique congruence was exhibited with respect to which the union curves coincide with the loxodromes. In particular, it was shown that the loxodromes of a sphere are union curves with respect to the congruence of lines which have the directions of the radii of the circles of latitude on the sphere.

7. *Pythagorean angles*, by Professor Arthur Bernhart, University of Oklahoma.

Consider the angles in a right triangle with integral sides. These are the angles whose sines, cosines and tangents are all rational, and half such angles have rational tangents; and conversely. The sum (or difference) of two Pythagorean angles is again Pythagorean, so that in any triangle if two angles are Pythagorean, the third is also. Half an angle is still Pythagorean if the subtended leg is even and the hypotenuse is an odd square.

Let $T_n(C_n, S_n)$ be an angle with rational tangent (cosine, sine) but whose other functions involve the square root of $n$. Then $T_m \pm T_n = T_{mn}$ as in the addition of logarithms. Similarly $C_n + C_n = C_n$, $C_n + S_n = S_n$, and $S_n + S_n = C_n$ as in the addition of even ($C$) and odd ($S$) integers.

An arbitrary triangle with integral sides has each angle of type $C_n$, indeed with the same $n$. The cosines are rational, but the sines, tangents, altitudes, inradius, circumradius and area are similar surds. Conversely if $C_n \pm C_{n'} = C_{n''}$, then $n = n' = n''$, and these are the angles of a triangle with commensurable sides. The type $C_1 = S_1 = T_1$ are the Pythagorean angles.

J. C. BRIXEY, *Secretary*

## NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Lehigh University, Bethlehem, Pennsylvania, on Saturday, November 25, 1950. In the absence of the Chairman of the Section, Professor G. C. Webber, University of Delaware, the meeting was conducted by the Secretary.

There were thirty-three present, including the following twenty-seven members of the Association: C. B. Allendoerfer, Laura M. Ashbaugh, Joshua Barlaz,

D. W. Blackett, R. K. Brown, J. O. Chellevold, F. E. Clark, C. D. Firestone, H. S. Grant, Theodore Hailperin, S. T. Hu, B. C. Kenny, P. A. Knedler, T. L. Koehler, V. V. Latshaw, L. J. Mordell, C. O. Oakley, W. B. Pitt, G. E. Raynor, H. A. Seebald, C. A. Shook, L. L. Smail, E. P. Starke, R. R. Stoll, Albert Wilansky, R. H. Wilson, Jr., H. J. Zimmerberg.

At the business meeting the following officers were elected for the coming year: Chairman, P. A. Caris, University of Pennsylvania; Secretary, C. O. Oakley, Haverford College. The Program Committee for the next meeting will be E. P. Starke (Chairman) Rutgers University, Alexander Tartler, Lafayette College, and A. W. Tucker, Princeton University. The 1951 meeting of the section will be held at the University of Pennsylvania on November 24.

The program consisted of the following papers:

1. *The essential roughness of mathematical objects*, by Professor Albert Wilansky, Lehigh University.

Given a set and a smoothness property of some of its elements, then most of the elements of the set do not have the property. A smoothness property is one such that if $f$ and $g$ have the property so does $f+g$, or $af+bg$. For example, continuity of functions, rationality of numbers. "Most" can be interpreted in such senses as category, denumerability.

Use is made of classical ideas such as theorems 2, p. 22, and 1, p. 36 of Banach's monograph, *Studia Math.* 1 (1929); p. 51 (Steinhaus), p. 92 (Mazurkiewicz), p. 174 (Banach), *Studia Math.* 3 (1931); art. 8.33, p. 173 of Zygmund's monograph, *Trigonometrical Series*; and also the ideas of Cantor. Some of these can be cast in the language used above. An extra condition (Baire property) on the "smooth subspaces" appears. This cannot be omitted as shown by an example, due to Hausdorff, of a proper subspace of a Banach space which is of the second category.

2. *Systems of axiomatic set theory*, by Professor C. D. Firestone, Rutgers University.

Among the various systems of mathematical logic which are adequate for all or part of classical mathematics, a prominent part is played by those known as systems of "axiomatic set theory." We consider three such systems: the Zermelo set theory ($Z$), the Zermelo-Skolem set theory ($ZS$), and the Godel set theory ($G$).

It is shown that $Z$ is inadequate for certain parts of classical mathematics, and it is indicated that $G$ is probably adequate for all of known classical mathematics. Further, it is proved that $Z$ is weaker than either $ZS$ or $G$ and that $ZS$ and $G$ are of equal strength. We consider also the possibility of constructing systems of set theory which are stronger than $G$ and a simple and natural procedure is indicated for constructing an infinite sequence of such systems each stronger than its predecessor.

3. *The coefficients of Schlicht functions*, by Professor Bernard Epstein, University of Pennsylvania (introduced by the secretary).

This paper was given by title. Let $S$ be the family of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which, in the circle $|z|<1$, are regular and schlicht—i.e., if $|z_1|<1$ and $|z_2|<2$ and $f(z_1)=f(z_2)$, then $z_1=z_2$. Similarly let $\Sigma$ be the family of all functions $g(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}$ meromorphic and schlicht for $|z|>1$. During the past three decades many very remarkable inequalities involving the coefficients $a_n$ and $b_n$ have been discovered. Many of them have been obtained by surprisingly elementary (but by no means obvious) methods. A basic result which has been frequently used is the "area-principle" of Bierberbach, namely $\sum_{n=1}^{\infty} n|b_n|^2 \leqq 1$. This leads rather easily to the inequality

$|a_2| \leqq 2$ and, less easily, to $|a_n| \leqq Kn$, $K$ an absolute (and as yet unknown) constant. Attempts to prove the Bieberbach conjecture that $K=1$ have led to the development of more profound methods. Among the most significant results obtained by such methods is that $|a_3| \leqq 3$, i.e., the aforementioned conjecture is correct for $n=3$.

More generally, the derivation of inequalities concerning the coefficients of functions schlicht (and suitably normalized) in a given domain has led to the solution of many interesting external problems closely connected with the theory of conformal mapping of simply and multiply connected domains.

4. *Topological properties of spaces of curves*, by Professor S. T. Hu, Institute for Advanced Study.

An elementary exposition was given of the most fundamental part of the works of M. Morse and E. Pitcher based on which recent developments have been made by various authors on topology as well as calculus of variations in the large.

A path in a given metric space $X$ with metric $d$ is a continuous map $\sigma: I \to X$ of the closed unit interval $I$. The totality of paths in $X$ form a metric space $W$ with the metric $d$ defined by

$$d(\sigma, \tau) = \sup_{t \in I} d[\sigma(t), \tau(t)].$$

The Frechet distance $d^*$ was also introduced in the space $W$, and its vanishing gives an equivalence relation in $W$. Thus the paths $W$ are divided into disjoint classes called the curves in $X$. By means of the Morse length and parametrization, a proof of Pitcher's theorem was sketched. This theorem states that the totality of the curves in $X$ with the Frechet distance form a metric space which is homeomorphic with a deformation retract of the space $W$.

C. O. OAKLEY, *Secretary*

## DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The regular fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Catholic University, Washington, D. C., on Saturday, December 9, 1950. Professor O. J. Ramler, Chairman of the Section, presided at both morning and afternoon sessions.

Ninety-seven people registered their attendance, including the following sixty-seven members of the Association: J. C. Abbott, R. P. Bailey, L. F. Boron, C. C. Bramble, B. H. Buikstra, W. E. Byrne, H. H. Campaigne, H. W. Charlesworth, H. J. Cheston, Randolph Church, G. R. Clements, Abraham Cohen, G. F. Cramer, J. A. Duerksen, F. D. Faulkner, W. J. Feeney, E. J. Finan, Joyce B. Friedman, A. M. Gleason, Michael Goldberg, R. A. Good, E. S. Grable, Clem Grabner, E. C. Gras, R. E. Greenwood, D. W. Hall, Marshall Hall, H. G. Hertz, S. B. Jackson, Walter Jennings, L. M. Kells, H. L. Kinsolving, A. E. Landry, B. J. Lockhart, C. G. Maple, M. H. Martin, J. M. McLynn, D. St. C. Melvin, Joseph Milkman, A. K. Mitchell, T. W. Moore, W. K. Morrill, H. Dorothy Mortell, Paul Nesbeda, M. W. Oliphant, Hyman Orlin, Walter Penney, F. M. Pulliam, O. J. Ramler, J. N. Rice, W. G. Rouleau, S. W. Saunders, E. D. Schell, W. T. Sharp, Sister Catherine Marie, E. R. Sleight, C. F. Stephens, H. C. Stotz, Feodor Theilheimer, O. M. Thomas, John Todd, Mary C. Varnhorn, C. H. Wheeler, P. M. Whitman, Mildred M. Wiker, J. W. Wrench, J. W. Wright.

It was voted to hold the next meeting on Saturday, April 28, 1951. This meeting will be at the United States Naval Academy in Annapolis, Maryland. Professor G. R. Clements, regional governor of the Section, reported on the meeting of the Board of Governors at Cambridge in September, during the International Congress. He commented on the medals available as awards for high school mathematics competition, the desire of the Association to reach a larger proportion of the Junior College teachers, and the attempts made to improve the teaching of mathematics in the schools. Mr. Ware Cattell described briefly a new scientific journal entitled *Research Report* devoted to prompt publication (fifteen to thirty days) of brief scientific articles (a thousand words or less). Mr. Cattell is the editor of this journal which is scheduled to begin publication early in 1951. A nominating committee was appointed consisting of Mr. Michael Goldberg, Professor E. J. Finan, and Professor D. W. Hall.

The program for the morning session consisted of four papers as follows.

1. *Finding the needle in the haystack*, by Dr. G. F. Cramer and Dr. H. H. Campaigne, Navy Department. (Presented by Dr. Campaigne)

The problem considered was to order a set of objects in such a way as to maximize the probability that an exceptional object (the needle) will be early in the order. The set of objects was taken to be continuous, and a formula was given for the first objects in the order. The ordering was in terms of a property of the objects. The special cases where this property is distributed normally and Poisson were considered.

2. *A summation process*, by Miss Barbara McGehee, University of Richmond, introduced by Professor E. R. Sleight.

In a paper by A. L. O'Toole entitled *Insights on Trick Methods*, National Mathematics Magazine, vol. XV, Oct. 1940, it was pointed out that if there exists a function $f(x)$ for which $u_x = f(x+1) - f(x)$, the series $\sum u_x$ from $x=a$ to $x=b$ is summed by the formula

$$\sum_{x=a}^{b} u_x = f(b+1) - f(a) = f(x)\Big]_a^{b+1}$$

This paper considered what types of series could be summed by this procedure. The method was used to sum arithmetic progressions, geometric progressions, series of the type $\sum n^k$ for positive integral $k$, and series $\sum 1/n(n+1)(n+3)$. It was conjectured that the method always applies when the $n$th term is a rational function of $n$.

3. *Isotopy and projective planes*, by Professor J. C. Abbott, United States Naval Academy.

Let $l$ be a fixed line of a projective plane $\pi$, 0 and $\infty$ fixed points on $l$, and $l_0$ and $l_\infty$ fixed lines through 0 and $\infty$ such that $l$, $l_0$, and $l_\infty$ are all distinct. Let $G$ be the set of points of $l$ distinct from 0 and $\infty$, $e$ any point in $G$, and $l_e$ any line through $e$ but not through the intersection of $l_0$ and $l_\infty$. Then under the classical definition of multiplication given by Veblen and Young, $G$ becomes a loop dependent on $l_e$ as a parameter, whereas principal isotopy is equivalent to the case that the change in parameter leaves its intersection with $l_\infty$ fixed.

4. *Connected subsets of a line*, by Professor D. W. Hall, University of Maryland.

A set $A$ was defined to be connected provided there exists no continuous mapping $f(A) = p+q$, where $p$ and $q$ are distinct points. This new definition of connectedness, which is easily seen to be equivalent to the usual one, yields very simple proofs of the standard theorems on connected sets. Connected subsets of the line are quickly characterized by this procedure.

5. *Effective processes and turing machines*, by Mr. W. W. Boone, Catholic University of America, introduced by the Secretary.

The program for the afternoon session consisted of this invited address by Mr. Boone. The paper was expository. The intuitive notion of an *effective process* for solving a problem was described as follows: a process which yields an answer to the problem in a finite number of steps, and, such that at no stage does the problem solver have a choice as to his next operation. The analogies of a machine and of concrete instructions furnished a clerk were discussed. The concern of algebraists and topologists—as well as logicians—for effective processes was emphasized. Relations between *effective enumerability* and effective processes were described. Various precise formalizations of the concept of an effective process, due to Church, Godel, Kleene, and Post were described. Post's concepts of *degree of unsolvability* and *creative set* were discussed, as were certain results of Kleene. The talk then proceeded with a detailed description of another technical representation of effective processes due to Turing, the *turing machine*. Proofs in the general theory of turing machines were sketched. Post's proof—using turing machines and identifying them with creative processes—that the word problem for semigroups is unsolvable was presented. Turing's analogous result for cancellation semigroups was described.

S. B. JACKSON, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Joint meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 5, 1951.
ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.
INDIANA, May 5, 1951.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, University of California, Berkeley, January 26, 1952.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.
PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
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UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.
WISCONSIN, Carroll College, Waukesha, May 12, 1951.

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